Function Smoothing with Applications to VLSI Layout*

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Abstract

We present approximations to non-smooth continuous functions by differentiable functions which are parameterized by a scalar $\beta > 0$ and have convenient limit behavior as $\beta \to 0$. For standard numerical methods, this translates into a tradeoff between solution quality and speed. We show the utility of our approximations for wirelength and delay estimations used by analytical placers for VLSI layout. Our approximations lead to more "solvable" problems.

1 Introduction

Continuous optimization problems often involve convex objective and constraint functions that are differentiable almost everywhere, but have non-differentiabilities due to directional derivatives disagreeing at some points. This non-differentiability can occur, e.g., when the functions involve absolute values. Examples in VLSI analytic placement include wirelength [1, 14] and delay [7, 8, 15], both of which depend on the absolute value of node-to-node distances. Examples abound in other applications, e.g., multifacility location [5, 11] and denoising in image processing.

Optimization methods that assume differentiability, e.g., Newton methods and variants [10, 18], are unsatisfactory if optima occur at or near points of non-differentiability.[1]

In recent applications [1, 5, 11], non-differentiability has been addressed by function *regularization*, i.e., removing non-differentiabilities without significantly changing the set of minimizers. Newton-type methods become applicable: their speed improves as the magnitude of the regularization increases, but optima of the regularized objective diverge from those of the original problem. To gauge this tradeoff, the regularization is parameterized by a scalar $\beta \geq 0$, with $\beta = 0$ corresponding to the original function and any $\beta > 0$ giving a smooth function amenable to numerical methods. Reasonable convergence properties as $\beta \to 0$ and problem-independent scaling of β allow the use of a regularized objective, instead of the original objective, for practical applications.

This paper proposes new generalized approaches to construct regularizations for given objectives, notably piecewise linear functions that are used by analytical placers in VLSI layout. A special case has been successfully applied to the minimization of linear wirelength [1] and gives a new interpretation of the well-known heuristic GORDIAN-L [14]. We give regularizations of linear wirelength and path-delay based objectives that cannot be produced by previous approaches.[2] Combining our proposed regularization with a novel strictly convex estimate for path-based delay yields problems that are amenable to Newton-type methods, yet are smaller and easier to solve than those produced by [7, 8, 15]. We thus achieve a new outlook on performance-driven analytical placement.

Section 2 describes our proposed methods of function regularization, complete with asymptotic theorems and typical examples. Section 3 discusses two issues critical for implementations: changes in the set of minimizers when the original objective is regularized, and scale-independent regularization. Applications to VLSI layout are given in Section 4, and conclusions are given in Section 5.

2 Function regularization

For an open subset $X \subset \mathbf{R^n}$, we assume a continuous convex function $f : X \to \mathbf{R}$ and seek a family of smooth convex functions $f_\beta(\cdot)$ for $\beta > 0$ such that

(a) $\lim_{\beta\to 0} f_\beta(\mathbf{x}) = f(\mathbf{x})$ uniformly on $\mathbf{R^n}$

(b) $\lim_{\beta\to 0} \inf_{\mathbf{x}\in\mathbf{R^n}} f_\beta(\mathbf{x}) = \inf_{\mathbf{x}\in\mathbf{R^n}} f(\mathbf{x})$

For simple functions, we provide "recipes" for regularization and prove their desired limit behaviour. For complicated functions, e.g., $f(x) = 2|x| + x^2$, we isolate non-differentiabilities to small symbolic fragments for which recipes exist. Replacing the symbolic fragments with their regularizations yields a regularization of the overall function.

2.1 Piecewise linear functions

We begin by considering $f : \mathbf{R} \to \mathbf{R}$ and distinguish a common case where regularizing f is easy:

$$f(x) = \begin{cases} \alpha_1(x - x_0) + C, & \text{if } x \geq x_0, \\ \alpha_2(x - x_0) + C, & \text{if } x < x_0, \end{cases} \qquad (1)$$

where $\alpha_1 > 0$, $\alpha_2 < 0$, and C is arbitrary.

For $p \geq 2$, the β-regularization of f is defined by:

$$\forall x, f_\beta(x) = C + (|f(x) - C|^p + \beta)^{\frac{1}{p}}. \qquad (2)$$

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[1]More sophisticated methods typically complicate algorithms, increase computational effort, and have convergence problems. E.g., sub-gradient optimization [9], use of an auxiliary variable and auxiliary inequality constraints [6, §4.2.3] or solution of a sequence of problems with updated weights in the objective function [6, §4.2.3].

[2]In [11], the l_p norm $(\sum |x_i|^p)^{\frac{1}{p}}$ is regularized with $(\sum |x_i|^p + \beta)^{\frac{1}{p}}$, which is not smooth for p=1 (the Manhattan norm which governs node-to-node distances used in wirelength and delay estimation).

This regularization is different from that in [11], as p is now a regularization parameter.

Example 1: If $f(x) = |x|$ and $p = 2$ then $f_\beta(x) = \sqrt{x^2 + \beta}$. This can be used to regularize the l_1-norm. The value $p = 2$ is typical since it is the smallest value for which the regularized function is twice-differentiable. (See Theorem 1.)

Theorem 1: $\forall p \geq 2, \forall\ \beta > 0$, f_β defined in (2) is:

(a) at least $(\lceil p \rceil - 1)$-times continuously differentiable,
(b) strictly convex,
(c) for $p > 2$ a non-integer: exactly $\lceil p \rceil - 1 = \lfloor p \rfloor$-times continuously differentiable,
(d) for $p \geq 2$ an integer: at least p-times differentiable (in fact, infinitely differentiable) *iff* $\alpha_1 = -\alpha_2$.

Proof: The function f_β is continuous everywhere and is infinitely-differentiable everywhere except at x_0. The second derivative exists and is positive everywhere, except possibly at x_0, leading to **(b)**.

The 1^{st},...,$(\lceil p \rceil - 1)^{th}$ left and right derivatives of $|f(x) - C|^p$ are all zero at x_0, while for p a non-integer its higher derivatives are infinite. The chain rule then implies **(a)** and **(c)**.

If $\alpha_1 = -\alpha_2$ and $p \geq 2$ is an integer, then $f_\beta(x) = C + ((\alpha_1)^p |x - x_0|^p + \beta)^{1/p}$, and is infinitely differentiable. If, instead, $\alpha_1 \neq -\alpha_2$ then the left and right p-th derivatives of $|f(x) - C|^p$ differ at x_0, proving the "only if" of **(d)**. □

Theorem 2: For f_β defined in (2), we have:

(a) $\forall x, |f_\beta(x) - f(x)| \leq \beta^{1/p}$,
(b) $\lim_{\beta \to 0} f_\beta(x) = \lim_{p \to \infty} f_\beta(x) = f(x)$ uniformly on $\mathbf{R}$,
(c) $\forall x, \forall \beta_1 > \beta_2 > 0$, $f_{\beta_1}(x) > f_{\beta_2}(x) > f_0(x) = f(x)$,
(d) $\lim_{\beta \to 0} \min_{x \in \mathbf{R}} f_\beta(x) = \min_{x \in \mathbf{R}} f(x)$,
(e) $\forall \beta > 0 \lim_{x \to \pm\infty} f_\beta(x) = f(x)$.

Proof: The inequalities **(c)** are shown by subtracting C and taking both sides of the inequality to the power p. Note that $f_\beta - C \geq \beta^{1/p} > 0$ for $\beta > 0$.

For **(a)**, observe that $\beta = |(f_\beta - C)^p - (f - C)^p| = |(f_\beta - C) - (f - C)| \cdot |(f_\beta - C)^{p-1} + \ldots + (f - C)^{p-1}| \geq (f_\beta - C)^{p-1} |f_\beta - f| \geq \beta^{\frac{p-1}{p}} |f_\beta - f|$.

Item **(b)** follows from **(a)**; **(d)** follows from **(b)** and **(c)**.

Using **(c)** and the above inequalities, we have $|f_\beta(x) - f(x)| \leq \frac{\beta}{(f_\beta(x) - C)^{p-1}}$. Since $\lim_{x \to \pm\infty} f_\beta(x) = \infty$ for $\alpha_1, \alpha_2 \neq 0$, we have that $\lim_{x \to \pm\infty} |f_\beta(x) - f(x)| = 0$, proving **(e)**. □

Note that 2**(e)** is not true anymore with $\alpha_1 = 0$ or $\alpha_2 = 0$ (e.g., $f(x) = \alpha_1 \max\{x, 0\}$ or $f(x) = -\alpha_2 \min\{x, 0\}$). These two cases can be reduced to $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ by rotating the plot around the coordinate center, which motivates an alternative regularization of f that is coordinate-independent. Consider the upper branch of a hyperbola with asymptotes going along the plot of $f(x)$. For $f(x) = \alpha_1 \max\{0, x - x_0\} + C$ (the same as (1) with $\alpha_2 = 0$) such a hyperbola can be defined in the x-y plane (i.e., for $y = f(x)$) with the following equation:[3]

$$(2(y - C)/\alpha_1 - x + x_0)^2 - (x - x_0)^2 = \beta. \tag{3}$$

[3]For $\alpha_1 > 0$ and $\alpha_2 < 0$, a hyperbola similar to that in Equation (3) would define an infinitely differentiable regularization, otherwise satisfying the statements of Theorems 1 and 2. For $\alpha_1 \neq -\alpha_2$, it will differ from the regularization defined in (1) because the latter is not twice-differentiable according to Theorem 1(d).

One can verify that Theorem 1 holds for this regularization with the regularized function being infinitely differentiable as well. Theorem 2 also holds. For piecewise linear functions, we need:

Fact 3: Any convex piecewise linear function with k linear segments can be presented (not necessarily uniquely) as a sum of $k - 1$ convex functions of the form (1), possibly with $\alpha_1 = 0$ or $\alpha_2 = 0$.

Corollary 4: Any convex piecewise linear function with k linear segments can be β-regularized with $|f(x) - f_\beta(x)| \leq \beta^{1/p}(k - 1)$. The regularization will possess properties from Theorems 1 and 2.

Example 2: If α_1 and α_2 in (1) are of the same sign, but $f(x)$ is still convex, then by Corollary 4 it can be regularized.

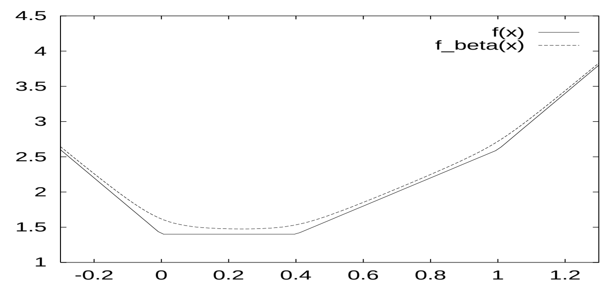

Figure 1: $f(x) = |1 - x| + 2|x| + |0.4 - x|$ with $p = 2$ and $f_\beta(x) = \sqrt{(1 - x)^2 + \beta} + 2\sqrt{x^2 + \beta} + \sqrt{(0.4 - x)^2 + \beta}$, $\beta = 0.01$.

2.2 Symbolic regularization and examples

For many functions, the cusps that need to be regularized are due to an absolute value or more general case analysis in the symbolic representation of the function. Define F by:

$$\forall x, F(x) = \begin{cases} F_1(x - x_0) + C, & \text{if } x \geq x_0, \\ F_2(x - x_0) + C, & \text{if } x < x_0, \end{cases} \tag{4}$$

with $F_1(t), t \leq 0$ and $F_2(t), t \geq 0$ continuously differentiable, non-negative, convex, and $F_1(0) = F_2(0) = 0$, but not necessarily $F_1'(0^+) = F_2'(0^-)$. Let C be arbitrary.

For $p \geq 2$, the β-regularization of $F(x)$ is defined by:

$$\forall x, F_\beta(x) = C + (|F(x) - C|^p + \beta)^{\frac{1}{p}}, \tag{5}$$

which subsumes (2) for the piecewise linear case.

Replacing a symbolic fragment with a regularization in a larger function leads to a smooth function.[4]

Example 3: $\max\{a, b\} = (a + b + |a - b|))/2$ and $\min\{a, b\} = (a + b - |a - b|))/2$ can be regularized as, respectively, $(a + b + (|a - b|^p + \beta)^{1/p})/2$ and $(a + b - (|a - b|^p + \beta)^{1/p})/2$.

In particular, for $f(x) = \max\{0, (x - x_0)\}$ and $p = 2$, $f_\beta(x) = \frac{1}{2}((x - x_0) + \sqrt{(x - x_0)^2 + \beta})$, which matches the hyperbola in (3) when $\alpha_2 = 1$.

Example 4: $f(a, b) = \max\{(a + b)^2, (a - b)^2\} = a^2 + b^2 + 2|ab|$ can be regularized as $f_\beta(a, b) = a^2 + b^2 + 2\sqrt{a^2 b^2 + \beta}$.

[4]The convexity properties and the limit behavior of the fragment regularizations often extend to the resulting function through sums, products, exponents, etc. Properties 1 and 2 from [11] provide excellent examples of such symbolic regularization; however, the non-differentiable fragments are regularized differently there.

3 Practical issues

When $f(x)$ is convex, but not strictly convex, it can have multiple minimizers. However, $f_\beta(x)$ is strictly convex for $\beta > 0$ and has only one minimizer (see, e.g., Figure 1). From the theorems in Section 2, under mild conditions a minimizer of f can be obtained as the limit of the minimizer of $f_\beta(x)$ as $\beta \to 0$. In some cases, the minimizer of $f_\beta(x)$ already minimizes $f(x)$, e.g., for any β the unconstrained minimizer of (2) is the unconstrained minimizer of (1).

Numerical methods using β-regularization require specific values of β to evaluate the regularization or its derivatives. Ideally, this should be independent of the scale of the arguments in the objective function. In other words, regularization should scale independently of β.

Proposition 5: Given $f(x) = |x|$, $\lim_{x\to\infty} f_\beta(kx)/f_\beta(x) = f(kx)/f(x), \ \forall\, k > 0\ \beta > 0$.
Proof: $\lim_{x\to\infty} k^p x^p + \beta/x^p + \beta = k^p$. □

To have $f_{\beta(x)}(kx)/f_{\beta(x)}(x) = Kf(kx)/f(x)$ for all x and some K, $\beta(x)$ must scale as x^p. In practice, the p^{th} exponent of the maximal x value for a problem can be multiplied by an instance-independent β_0 to produce β.

4 Applications to Analytical Placement

4.1 Wirelength approximation

Analytical placers position nodes to minimize wirelength by solving a sequence of optimization problems. Since exact wiring of edges is unknown, linear wirelength estimates are used. Typically, linear constraints of the form $\mathbf{Hx} = \mathbf{b}$ are included. The resulting problem:

$$\min_{\mathbf{x}}\{\textstyle\sum_{i>j} a_{ij}|x_i - x_j| : \mathbf{Hx} = \mathbf{b}\} \qquad (6)$$

($\mathbf{x}$ represents unknown node positions) is not amenable to Newton-type methods since it is neither differentiable nor strictly convex. However, since the number of unknowns $\mathbf{x}$ is large (10^4 – 10^6), Newton-type methods are essential for computational efficiency.

One methodology, GORDIAN-L [14], is based on updating weights on the objective [6, §4.2.3] and considers:

$$\min_{\mathbf{x}^\nu}\{\textstyle\sum_{i>j} \frac{a_{ij}}{|x_i^{\nu-1} - x_j^{\nu-1}|}(x_i^\nu - x_j^\nu)^2 : \mathbf{Hx}^\nu = \mathbf{b}\}, \qquad (7)$$

where $\mathbf{x}^{\nu-1}$ and $\mathbf{x}^\nu$ denote the vectors of node positions at iterations $\nu - 1$ and ν. A quadratic objective is used to avoid the non-differentiability of (6), but the coefficients of the objective are updated iteratively to approximate the linear wirelength estimate.

As an alternative, regularization of (6) has been proposed in [1] and considers:

$$\min_{\mathbf{x}}\{\textstyle\sum_{i>j} a_{ij}\sqrt{(x_i - x_j)^2 + \beta} : \mathbf{Hx} = \mathbf{b}\}. \qquad (8)$$

This optimization problem was solved in [1] in two ways: with a linearly-convergent fixed-point method due to Eckardt's [2, 3] generalization of the Weiszfeld algorithm [17], and with a novel primal-dual Newton method having quadratic convergence. Numerical testing in [1] illustrates the tradeoffs in values of $\beta > 0$ versus time and difficulty.[5]

[5] It is also shown in [1] that the GORDIAN-L heuristic can be interpreted as a special case $\beta = 0$ of a fixed-point method having guaranteed linear convergence for $\beta > 0$. This can be seen by differentiating the objective function in (8), setting $\beta = 0$ and comparing to (7).

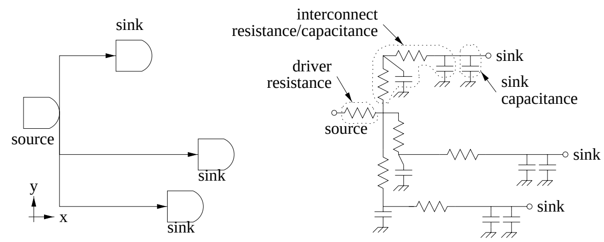

Figure 2: Wiring model for a single net.

4.2 Delay approximation

Performance-driven analytical placers typically rely on Elmore delays [4] in approximating critical path delays. If an equivalent-Π model is used for each segment of interconnect (i.e., a lumped-distributed model with half the capacitance at each end), then the Elmore delay is a *posynomial*, but not strictly convex, function of the lengths of the interconnect segments.[6] Not all objectives and constraints can be cast as posynomials, e.g., absolute values and node positions appearing in length calculations (cf. (7)).

A popular *ad hoc* approach is to convert timing analysis results into net weights [16, 13] incorporated into the wirelength objective function. Path-based delays have also been included explicitly as nondifferentiable constraints [7, 8, 15], but numerical solutions are hard and the constraints are many, e.g., for k critical paths, each with an average of e edges, ke constraints are required in [8].

Thus, the problem of including explicit performance information is still open. Two elements are necessary: (i) strictly convex delay estimates in terms of node positions and (ii) their regularization to remove non-differentiabilities.

For a given net we use rectilinear L-shaped interconnects to connect the source directly to each sink (see Figure 2). An equivalent-L model is used for each of the two segments of each L-shaped interconnect. Let C_s be the capacitance of sink s, R_d be the driver resistance and r_x (r_y) and c_x (c_y) be the per-unit interconnect series resistance and shunt capacitance, resp., in the x- (y-) direction equivalent-L model. The delay from the source to a specific sink consists of:

(a) Source resistance times all downstream capacitance:
$$R_d(\textstyle\sum_s c_x|\alpha| + c_y|\gamma| + C_s) \qquad (9)$$

(b) Interconnect resistance times sink capacitance:
$$(r_x|\alpha| + r_y|\gamma|)C_s \qquad (10)$$

(c) Interconnect resistance times interconnect capacitance:
$$r_x c_x \alpha^2 + r_y c_y \gamma^2 + r_y c_x |\alpha||\gamma|. \qquad (11)$$

where $\alpha = x_d - x_s$ and $\gamma = y_d - y_s$. Components **(a)** and **(b)** are convex. However, they are non-differentiable when nodes are aligned vertically or horizontally. Component **(c)** is clearly convex if (but not "only if") the cross term $|\alpha||\gamma|$ is ignored. The magnitude of the cross term can be comparable to other terms and should not be ignored if strict convexity can be otherwise guaranteed.

Proposition 6: If $c_y/r_y > 0.25c_x/r_x$ then the delay component **(c)** is strictly convex.
Proof: The functions $q_-, q_+ : \mathbf{R}^2 \to \mathbf{R}$

$$q_-(\alpha,\gamma) = \alpha^2 r_x c_x + \gamma^2 r_y c_y - r_y c_x \alpha\gamma, \qquad (12)$$
$$q_+(\alpha,\gamma) = \alpha^2 r_x c_x + \gamma^2 r_y c_y + r_y c_x \alpha\gamma, \qquad (13)$$

[6] If all objective and constraint functions are posynomial functions, then a transformation can be used to produce a convex problem with strictly convex objective [6, §6.8.2.3].

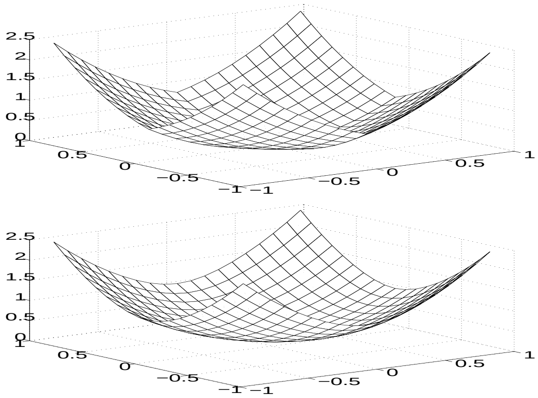

Figure 3: Regularization of the interconnect delay.

are strictly convex under the assumption of the proposition. The function $d(\alpha,\gamma) = \max\{q_-(\alpha,\gamma), q_+(\alpha,\gamma)\}$ is the maximum of two strictly convex functions and is hence strictly convex. The function $d(x_d - x_s, y_d - y_s)$ is strictly convex due to the composition and equals the interconnect delay. □

Thus, given suitable capacitance to resistance ratios in the different routing directions (e.g., different metal layers), the cross term can be kept. Regularization of delay components **(a)** and **(b)** is straightforward. Component **(c)** is regularized similarly to Example 4 in Section 2.2 as

$$\sqrt{(\alpha\gamma r_y c_x)^2 + \beta} + \alpha^2 r_x c_x + \gamma^2 r_y c_y. \qquad (14)$$

Figure 3 illustrates (11) and its regularization (14).

Finally, critical path delays can be taken as the sum of the delay along those edges along the path. Since the individual edge delays are convex, so is the path delay.

4.3 Wire length and delay brought together

We propose several concise performance-driven formulations. For penalization of critical path delays, we propose

$$\min_{x,y}\{f(x,y) + K \textstyle\sum_{\pi\in P} d^{\pi}_{\beta}(x,y) : x,y \in \Omega\}, \qquad (15)$$

where $f(x,y)$ is the wirelength estimate, P is the set of critical paths, $d^{\pi}_{\beta}(x,y)$ is the regularized delay for path π, scalar K normalizes the delay and wirelength terms and Ω represents the set of constraints.

For minimization of longest path delays we propose

$$\min_{x,y}\{f(x,y) + K \textstyle\sum_{\pi\in P} \max\left\{\zeta, d^{\pi}_{\beta}(x,y)\right\} : x,y \in \Omega\}, \quad (16)$$

where ζ is a "soft" target delay. Since delay information is included directly in the objective function, the number of constraints is *not increased* by the inclusion of performance-driven information.

Finally, for "hard" timing constraints we propose:

$$\min_{x,y}\left\{f(x,y) : d^{\pi}_{\beta}(x,y) \le \zeta, \quad \forall \pi \in P \ : \ x,y \in \Omega\right\}. \quad (17)$$

Regularization results in one constraint per critical path and still allows for Newton-type methods (e.g., [18]).

5 Conclusions

Numerical solvers perform best with smooth and convex functions. Non-differentiable points often arise when, e.g., Manhattan distances, are used. We have presented general, provably good regularization techniques for eliminating cusps in optimization objectives, and have discussed their theoretical and practical properties. Our techniques are more general and "modular" than those previously proposed and can be applied to large classes of functions. Their utility has been demonstrated for wirelength- and delay-based objectives in VLSI applications, where they lead to smaller and easier performance-driven placement formulations.

References

[1] C. J. Alpert, T. F. Chan, D. J. H. Huang, A. B. Kahng, I. L. Markov, P. Mulet and K. Yan, "Faster Minimization of Linear Wire Length for Global Placement", *Proc. ACM/IEEE Intl. Symp. on Physical Design*, 1997, pp. 4-11.

[2] U. Eckhardt, "On an Optimization Problem Related to Minimal Surfaces with Obstacles", in R. Bulirsch, W. Oetti and J. Stoer, eds., *Optimization and Optimal Control*, *Lecture Notes in Mathematics* 477, Springer Verlag, 1975, pp. 95-101.

[3] U.Eckhardt, "Weber's Problem and Weiszfeld's Algorithm in General Spaces", *Mathematical Programming* 18 (1980), pp. 186-196.

[4] W. C. Elmore, "The Transient Response of Damped Linear Networks with Particular Regard to Wideband Amplifiers", *J. Applied Physics*, 19 (1948), pp. 55-63.

[5] J.W. Eyster, J.A. White and W.W. Wierwille, "On Solving Multifacility Location Problems Using a Hyperboloid Approximation Procedure", *Trans. American Institute of Industrial Engineering* 5 (1973), pp. 1-6.

[6] P.E. Gill, W. Murray and M.H. Wright, *Practical Optimization*, 1981, Academic Press, London.

[7] M. A. B. Jackson and E. S. Kuh, "Performance-Driven Placement of Cell Based IC's", *Proc. ACM/IEEE Design Automation Conf.*, 1989, pp. 370-375.

[8] T. Hamada, C. K. Cheng and P. M. Chau, "Prime: A Timing-Driven Placement Tool Using a Piecewise Linear Resistive Network Approach", *Proc. ACM/IEEE Design Automation Conf.*, 1991, pp. 531-536.

[9] M. Held, P. Wolfe and H. P. Crowder, "Validation of Subgradient Optimization", *Mathematical Programming*, 1974, 6, pp. 62–88

[10] C. Kelley, *Iterative Methods for Linear and Nonlinear Equations*, *Frontiers in Applied Mathematics* vol. 16, SIAM Press, 1995.

[11] R.F. Love and J.G. Morris, "Solving Constrained Multifacility Location Problems Involving l_p Distances Using Convex Programming", *Operations Research* 23 (1975), pp. 581-587.

[12] I. I. Mahmoud, K. Asakura, T. Nishibu and T. Ohtsuki, "Experimental Appraisal of Linear and Quadratic Objective Functions Effect on Force Directed Method for Analog Placement", *IEICE Trans. on Fundamentals of Electronics, Communications and Computer Sciences* 4(E77-A) (1994), pp. 710-725.

[13] B. M. Riess and G. G. Ettelt, "Speed: Fast and Efficient Timing Driven Placement", *Proc. IEEE Intl. Symp. on Circuits and Systems*, 1995, pp. 377-380.

[14] G. Sigl, K. Doll and F. M. Johannes, "Analytical Placement: A Linear or Quadratic Objective Function?" *Proc. ACM/IEEE Design Automation Conf.*, 1991, pp. 57-62.

[15] A. Srinivasan, K. Chaudhary and E. S. Kuh, "RITUAL: A Performance Driven Placement for Small-Cell ICs", *Proc. Intl. Conf. on Computer-Aided Design*, 1991, pp. 48-51.

[16] R. S. Tsay and J. Koehl, "An Analytical Net Weighting Approach for Performance Optimization in Circuit Placement", *Proc. ACM/IEEE Design Automation Conf.*, 1991, pp. 620-625.

[17] E. Weiszfeld, "Sur le Point pour Lequel la Somme des Distances de n Points Données est Minimum." *Tôhoku Mathematics J.* 43 (1937), pp. 355-386.

[18] S.J. Wright, *Primal-Dual Interior-Point Methods*, Philadelphia, SIAM Press, 1996.