Analysis of RC Interconnections Under Ramp Input *

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Abstract

We present a general and, in the limit, *exact* approach to compute the time-domain response for finite-length *RC* lines under ramp input, by summing distinct diffusions starting at either end of the line. We also obtain analytical expressions for the finite time-domain voltage response for an open-ended finite *RC* line and for a finite *RC* line with capacitive load. Delay estimates using our new method are very close to SPICE-computed delays. Finally, we present a general recursive equation for computing the higher-order diffusion components due to reflections at the source and load ends. Future work extends our method to response computations in general interconnection trees by modeling both reflection and transmission coefficients at discontinuities.

1 Introduction

Estimating delays on VLSI interconnects is a key element in timing verification, gate-level simulation and performance-driven layout design. Because of their highly resistive nature, interconnects are generally modeled as distributed *RC* lines. The analysis of finite *RC* transmission lines with step input is widely discussed in the literature, e.g., [20, 1, 19]. The standard approach is to first calculate the transfer function; then, by approximating the transfer function both transform-domain and time-domain responses are obtained for different configurations of the finite *RC* line with step input. Using different approaches to invert the Laplace transform of the response, [16, 12, 13, 18] have all obtained the exact time-domain response for a finite-length open-ended *RC* line. The most recent of these works, by Rao [18], also extends the traditional transform-domain analysis to calculate the time-domain response for a finite *RC* line with capacitive load impedance. A direct solution of the open-ended finite *RC* line response, i.e., directly in the time-domain as an infinite series, was first given by Kaufman and Garrett [10]. Kahng and Muddu [12, 13] calculated the time-domain response in a finite distributed *RC* line with source and load impedances; the total response was shown to be equal to an infinite sum of diffusion equation solutions with each diffusion starting at either the source or load end of the line.

None of these previous works gives an understanding of the interconnect response when the *input signal* has nonzero transition time. It is more reasonable to model the input signal from driver to interconnect as a finite ramp. Figure 1 shows the substantial difference in the response for step versus ramp input. Kaupp [9] analyzed *RC* interconnections under finite ramp input by assuming infinitely long transmission lines. Extending this work, [3] approximated the transfer function of a semi-infinite line using a linear function, and proposed a model for *RC* lines under ramp input. Recently, [17] proposed a methodology for *RC* interconnect synthesis under ramp input using the first few moments of the transfer function. However, no analytical solution for the time-domain finite ramp response of a finite distributed *RC* line with source and load impedances under finite ramp input has been obtained in the literature.[1]

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[1]The definition of Elmore delay for step input [7] can be used to compute the analytical delay formula under ramp input, i.e., $T_D = \frac{1}{V_0}\int_0^\infty t v'_{out}(t)dt$ where $v'_{out}(t)$ is the derivative of the output response under finite ramp input. This definition implies that the ramp input delay is equal to the first moment of the derivative of the response. In the transform domain, this implies that the ramp input delay is equal to the first moment (or coefficient of s) of the function $s \cdot V_{out}(s)$. Therefore, an analytical expression for ramp input delay is

$$T_D = \frac{T_R}{2} + b_1 - a_1 = \frac{T_R}{2} + T_{ED}$$

where T_R is the rise-time of the ramp input and T_{ED} is equal to the Elmore delay for a step input or the first moment of the transfer function. Other analytical definitions of delay such as, $\int_0^\infty (1 - v_{out}(t))dt$ [15] lead to the same expression for delay.

In this paper, we analyze finite distributed *RC* lines under ramp input, using a new technique based on solving the diffusion equation and applying the method of images [12, 13]. Using this new technique, we are able to analytically obtain the transient time-domain response of a finite *RC* line for different cases of source and load impedances. Our contributions are the following.

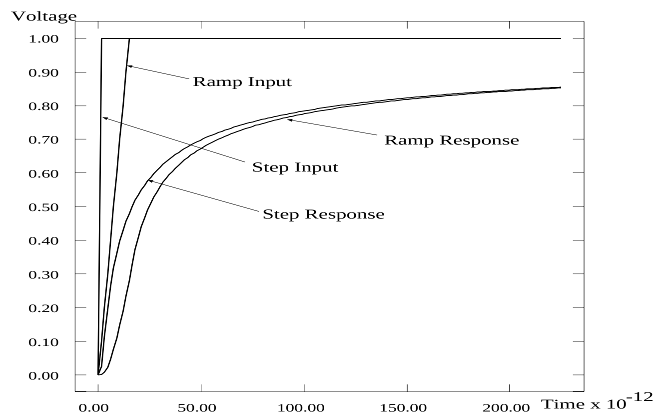

Figure 1: Response of semi-infinite line with step and finite ramp inputs. Rise-time and position time constant $R_x C_x$ are both 15 ps.

- We obtain the time-domain response of a semi-infinite *RC* line for both infinite and finite ramp inputs by solving the diffusion equation with appropriate boundary conditions. This result matches that given in [9].
- We provide a general approach to compute the time-domain response for finite *RC* lines by summing distinct diffusions which each start at an end of the line and can be viewed as traveling either forward or backward along the line in analogy with reflections. In the limit, this approach is exact; only a few reflections are needed to achieve accurate response computations.
- We obtain the analytical expressions for the time-domain voltage response under ramp input for an open-ended finite *RC* line and for a finite *RC* line with capacitive load. To the best of our knowledge, there is no previous literature on this subject. We compare delay estimates from our approach and from SPICE with URC (Uniform Distributed *RC*) model for *RC* lines: using only the first few reflected diffusion components in the voltage response, our delay estimates are very close to SPICE-computed delays. Finally, we present a general recursive equation for computing the higher order diffusion components due to reflections at the source and load ends of the interconnect line.

2 Semi-Infinite *RC* Line Analysis

Consider the semi-infinite distributed *RC* line shown in Figure 2. The voltage and current on a uniform distributed *RC* line are governed by the diffusion equation

$$rc\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} \quad (1)$$

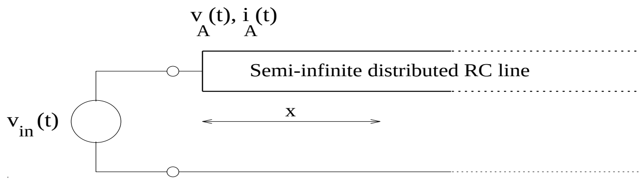

Figure 2: Semi-infinite distributed *RC* line, and position x on the line.

where r and c are resistance and capacitance per unit length. The solution to the diffusion equation under various boundary conditions has been well studied [11]. The work of [12] showed that the time-domain response of a finite-length *RC* line with step input was equal to an infinite sum of diffusion equation solutions, with each diffusion starting at either the source or the load end of the line.

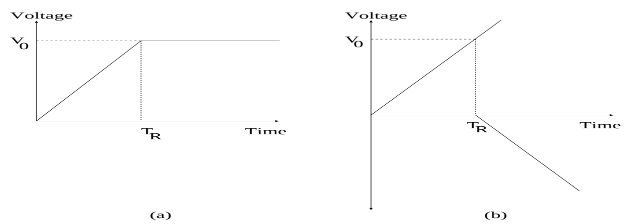

Figure 3: A ramp input function: (a) finite ramp with rise time T_R, and (b) finite ramp decomposed into two shifted infinite ramps.

We wish to calculate the response for a finite *RC* line under finite ramp input. We will first solve the above diffusion equation for the semi-infinite line, then represent the total voltage response on the finite line as the sum of *incident* and *reflected diffusion components*. The initial and boundary conditions for the semi-infinite line under finite ramp input (see Figure 3) are

$$\begin{array}{llll} \text{IC:} & v(x,0) & = & 0\ ,\ x \geq 0 \\ \text{BC1:} & v(0,t) & = & v_{in}(t) = \frac{V_0}{T_R}\left[tU(t) - (t-T_R)U(t-T_R)\right]\ ,\ t \geq 0 \end{array}$$

where T_R is the rise-time of the finite ramp input and $U(t)$ denotes the step function. We will also consider an infinite ramp input since any finite ramp can be expressed as the sum of two shifted infinite ramps (see Figure 3); the time-domain response for a finite ramp can be derived from the infinite ramp response by a change of time variable. Using $u(x,t)$ to represent the response for an infinite ramp input, the diffusion equation and new boundary conditions are:[2]

$$rc\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\begin{array}{lllll} \text{IC:} & u(x,0) & = & 0 & \text{for all } x \geq 0 \\ \text{BC1:} & u(0,t) & = & \frac{V_0}{T_R}\cdot t \cdot U(t) & \text{for all } t \geq 0 \end{array}$$

The diffusion equation for step input has a boundary condition that is constant with respect to time. For a ramp input this boundary condition is a function of time, so it is difficult to derive the solution in the same way as for a step input. However, differentiating the diffusion equation in time and using the variable $w(x,t) = \frac{\partial u(x,t)}{\partial t}$, we again obtain a diffusion equation

$$rc\frac{\partial w(x,t)}{\partial t} = \frac{\partial^2 w(x,t)}{\partial x^2}$$

[2]In the transform and time domains, we respectively use $U(x,s)$ and $u(x,t)$ to indicate the response for the infinite ramp input, and $V(x,s)$ and $v(x,t)$ to indicate the response for the finite ramp input.

with initial and boundary conditions obtained by taking the time derivative of the boundary conditions of $u(x,t)$. The initial condition remains the same, but the boundary condition for the new diffusion equation becomes constant in time, i.e., similar to that for a step input:

$$\begin{array}{lllll} \text{IC:} & w(x,0) & = & 0 & \text{for all } x \geq 0 \\ \text{BC1:} & w(0,t) & = & \frac{V_0}{T_R}U(t) & \text{for all } t \geq 0 \end{array}$$

The solution for the diffusion equation under step input can be obtained using the parabolic substitution of the variable $\eta = x\sqrt{\frac{rc}{2t}}$ [12] as

$$w(\eta) = C_1\int_0^{\eta} e^{-\frac{y^2}{2}}dy + C_2$$

The initial condition IC implies $C_1 = -\sqrt{\frac{2}{\pi}}C_2$. The boundary condition BC1, that at position $x=0$ the derivative of voltage is constant and equal to $\frac{V_0}{T_R}$ for all $t>0$, implies $C_2 = V_0$. Therefore,

$$w(x,t) = \frac{V_0}{T_R}[1 - erf(\frac{\eta}{\sqrt{2}})] = \frac{V_0}{T_R}\ erfc\left(\frac{b}{\sqrt{4t}}\right) \tag{2}$$

where x is the position at which the response is calculated, and $b = \sqrt{R_xC_x} = x\sqrt{rc}$. From this, the *incident diffusion component* $u_I(x,t)$ for the semi-infinite *RC* line under infinite ramp input can be derived as [14][3]

$$\begin{aligned} u_I(x,t) &= \int_{\tau=0}^{\tau=t} w(x,\tau)d\tau = \int_{\tau=0}^{\tau=t}\frac{V_0}{T_R}\ erfc\left(x\sqrt{\frac{rc}{4\tau}}\right)d\tau \\ &= \frac{V_0}{T_R}\left[(t+\frac{b^2}{2})erfc(\frac{b}{\sqrt{4t}}) - b\sqrt{\frac{t}{\pi}}e^{-\frac{b^2}{4t}}\right]U(t) \\ &= \frac{V_0}{T_R}\left[(t+\frac{R_xC_x}{2})erfc(\sqrt{\frac{R_xC_x}{4t}}) \right. \\ &\quad \left. -\sqrt{\frac{R_xC_xt}{\pi}}e^{-\frac{R_xC_x}{4t}}\right]U(t) \end{aligned} \tag{3}$$

Then, the time-domain response for the incident diffusion component $v_I(x,t)$ with a finite ramp input can be written in terms of the infinite ramp response:[4]

$$\begin{aligned} v_I(x,t) &= u_I(x,t) - u_I(x,t-T_R) \\ &= \frac{V_0}{T_R}\left[(t+\frac{b^2}{2})erfc(\frac{b}{\sqrt{4t}}) - b\sqrt{\frac{t}{\pi}}e^{-\frac{b^2}{4t}}\right]U(t) \\ &\quad - \frac{V_0}{T_R}\left[(t-T_R+\frac{b^2}{2})erfc(\frac{b}{\sqrt{4(t-T_R)}}) \right. \\ &\quad \left. +b\sqrt{\frac{(t-T_R)}{\pi}}e^{-\frac{b^2}{4(t-T_R)}}\right]U(t-T_R) \end{aligned} \tag{4}$$

As expected, the second term in the above equation is zero for $t \leq T_R$, so the finite ramp response is given by Equation (3) for $t \leq T_R$ and by Equation (4) for $t \geq T_R$.[5] The above analytical expressions for the ramp input response allow direct and efficient computation of delay estimates for ramp and piecewise-linear inputs.

[3]We can also compute the response for ramp input by using the transfer function of the semi-infinite *RC* line, $H_I(x,s) = e^{-x\sqrt{rcs}}$, and the infinite ramp input, $U_{in}(s) = \frac{V_0}{T_R}\cdot\frac{1}{s^2}$. The incident diffusion component in the transform domain is $U_I(x,s) = \frac{V_0}{T_R s^2}\cdot e^{-\sqrt{R_xC_xs}}$. The time-domain response obtained by applying the inverse Laplace transform [6] is exactly equal to the response given in Equation (3).

[4]The response in the transform domain for the incident diffusion component is $V_I(x,s) = \frac{V_0}{T_R s^2}(1-e^{-sT_R})e^{-b\sqrt{s}}$.

[5]Recall that the response for the semi-infinite line under step input [12] is $v_I(x,t) = V_0 erfc(\frac{b}{\sqrt{4t}})$.

3 Finite *RC* Line Analysis

The voltage at the front end of the line (i.e, at A) is

$$V_A(s) = V_{in}(s) \cdot \frac{Z_0}{Z_0 + Z_S} = V_{in}(s) \cdot \frac{(1 - \Gamma_S(s))}{2}$$

where $\Gamma_S(s) = \frac{Z_S - Z_0}{Z_s + Z_0}$ is the reflection coefficient at the source. For a general finite *RC* line with source and load impedance as shown in Figure 4, the incident propagation of voltage in the transform domain is

$$V_I(x,s) = V_A(s) e^{-\sqrt{R_x C_x s}} = V_{in}(s) \frac{(1 - \Gamma_S(s))}{2} e^{-\sqrt{R_x C_x s}}$$

The total voltage for a finite line (Figure 4) is the summation of the incident diffusion component and *reflected diffusion* components that arise at the source (S) and load (L) discontinuities. In other words, the time-domain expansion for total voltage is $v_{Tot}(x,t) = v_I(x,t) + \sum_{i=1}^{\infty} v_{R_i}(x,t)$ where $v_I(x,t) \equiv$ voltage due to the incident diffusion and $v_{R_i}(x,t) \equiv$ voltage due to the i^{th} reflection.[6] (In our notation, R_i refers to the i^{th} *reflected diffusion* starting from either the source or the load discontinuity; i basically represents the number of trips up and down the line.) In general, $v_{R_i}(x,t)$ can be calculated through convolution of the reflected diffusion (taking into account position displacement) with the reflection coefficients $\Gamma_S(t)$ or $\Gamma_L(t)$.

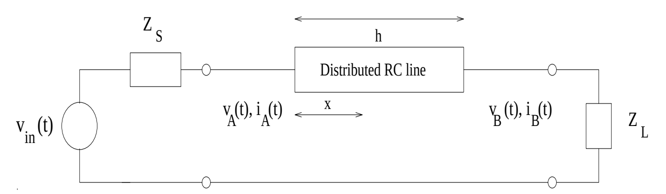

Figure 4: A distributed *RC* line of length h, and position x along the line.

The reflection coefficient at the source in the transform domain is $\Gamma_S(s) = \frac{Z_S - Z_0}{Z_S + Z_0}$, and the reflection coefficient at the load is $\Gamma_L(s) = \frac{Z_L - Z_0}{Z_L + Z_0}$. As shown in [13], the voltage at the position x in Figure 4 due to the first reflection at the load can be calculated from the incident wave and shifting in position by $h + h - x = 2h - x$, i.e., $V_{R_1}(x,s) = \Gamma_L(s) V_I(2h - x, s)$. The corresponding time-domain expression is $v_{R_1}(x,t) = \int_{\tau=0}^{t} \Gamma_L(t - \tau) v_I(2h - x, \tau)\, d\tau$, i.e., the first reflected voltage travels distance h to the end of the line before reflection, then additional distance $h - x$ to reach the specified location. Another explanation for the reflection voltages is by applying the symmetry argument in the Method of Images (or Reflections) [11, 13] to satisfy the boundary condition at the end of the line $x = h$. The total voltage on the line can also be proved to be equal to the sum of incident and reflected diffusion components by considering the response obtained from the 2-port transfer function of the line [13]. The total voltage can be expressed in the transform domain as a summation of various reflected components

$$\begin{aligned} V_{Tot}(x,s) &= V_I(x,s) + \sum_{n=1}^{\infty} \left[\Gamma_L^n(s) \Gamma_S^{n-1} V_I(2nh - x, s) \right. \\ &\quad \left. + \Gamma_L^n(s) \Gamma_S^n(s) V_I(2nh + x, s) \right] \end{aligned} \quad (5)$$

and the time-domain response is

$$\begin{aligned} v_{Tot}(x,t) &= v_I(x,t) + \sum_{n=1}^{\infty} \left[\int_{\tau=0}^{t} a_n(t-\tau) v_I(2nh - x, \tau)\, d\tau \right. \\ &\quad \left. + \int_{\tau=0}^{t} b_n(t-\tau) v_I(2nh + x, \tau)\, d\tau \right] \end{aligned} \quad (6)$$

[6] Similarly, the total voltage in the transform domain is $V_{Tot}(x,s) = V_I(x,s) + \sum_{i=1}^{\infty} V_{R_i}(x,s)$.

where $a_n(t)$ and $b_n(t)$ represent odd and even n^{th} reflection coefficient values. This methodology allows us to compute the response under ramp input for various cases of the finite-length line. The total response can be approximated by considering the analytical expressions for the first few reflection components; for higher accuracy, additional terms can be incorporated by using numerical techniques.

4 Open-Ended Finite *RC* Line Analysis

A finite open-ended line with ideal source has $\Gamma_S(s) = -1$ and $\Gamma_L(s) = 1$. The time-domain incident diffusion component for infinite and finite ramp input is given by Equations (3) and (4). Since all the coefficients of reflection are constants for an open-ended line with an ideal source, the total response can be computed from the incident diffusion component via shifts in the time variable, i.e., the time-domain response is

$$v_{Tot}(x,t) = v_I(x,t) + \sum_{n=1}^{\infty} \left[(-1)^{n-1} v_I(2nh - x, t) + (-1)^n v_I(2nh + x, t) \right]$$

where $v_I(x,t)$ is the incident diffusion component for finite ramp input given in Equation (4). Figure 5 compares the voltage response at the end of the line between SPICE and an approximation, called **Diff4**, which sums only up to the first four reflected diffusion components.

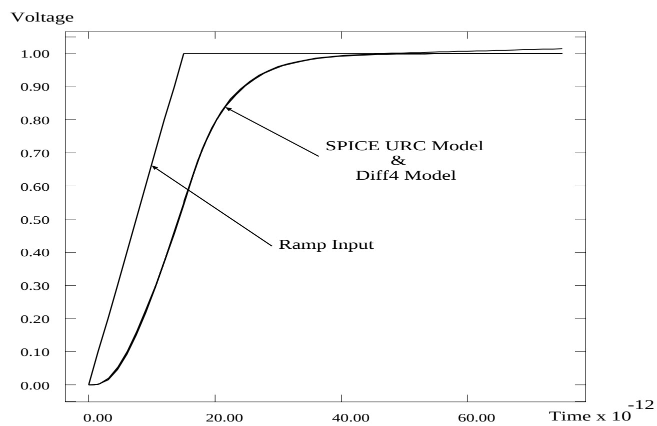

Figure 5: Response at the end of an open-ended line for a finite ramp with rise time $T_R = R_h C_h$, using SPICE and an approximation which sums up to the first four reflected diffusion components. The line parameters are $r = 0.015\ \Omega/\mu m$, $c = 0.25\ fF/\mu m$ and length $h = 2000\mu m$. The time constant of the line is $R_h C_h = 15.0$ ps.

5 Finite *RC* Line with Capacitive Load

A finite distributed *RC* line of length h with capacitive load C_L at the end of the line is shown in Figure 6. Recall from Equation (6) that the total response on the line is obtained by summing an infinite series of diffusion components due to reflections at the load and source. We now review the calculation of up to the first four reflected components (some details must be omitted for space reasons). If desired (e.g., for larger loads than those we consider), more reflection components can be calculated using methods given in Section 6.

Incident Diffusion. The first component of the total response is the incident diffusion voltage, derived in Equations (3) and (4). Diffusion components for reflections are computed by multiplying the reflection coefficients with the incident diffusion response in the transform domain as described in Section 3.

First Reflection. The reflection coefficient at the load for a load impedance of $Z_L = \frac{1}{sC_L}$ is $\Gamma_L(s) = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1 - q\sqrt{s}}{1 + q\sqrt{s}}$ where $q = C_L \sqrt{\frac{R_h}{C_h}} = \frac{C_L}{C_h} \sqrt{R_h C_h}$. The voltage response of the first reflected diffusion can be

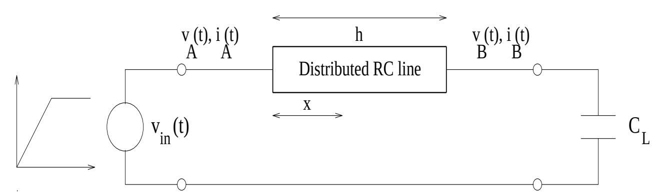

Figure 6: Distributed RC line of length h with capacitive load C_L.

obtained from Equation (6):

$$\begin{aligned} V_{R_1}(x,s) &= \Gamma_L V_I(2h-x,s) \\ &= \frac{V_0}{T_R s^2}(1-e^{-sT_R})\frac{1-q\sqrt{s}}{1+q\sqrt{s}}e^{-\frac{(2h-x)}{h}\sqrt{sR_hC_h}} \end{aligned}$$

The infinite ramp input response is

$$U_{R_1}(x,s) = \frac{V_0}{T_R s^2}\frac{1-q\sqrt{s}}{1+q\sqrt{s}}e^{-b\sqrt{s}}$$

where $b = \frac{(2h-x)}{h}\sqrt{R_hC_h}$. To compute the time-domain response we express the response in the transform domain in the form of $\frac{F_1(\sqrt{s})}{\sqrt{s}}$. Let

$$\begin{aligned} F_1(s) &= \frac{V_0}{T_R s^3}\frac{1-qs}{1+qs}e^{-bs} \\ &= \frac{V_0}{T_R}\left[\frac{1}{s^3}-\frac{2q}{s^2}+\frac{2q^2}{s}-\frac{2q^2}{(s+1/q)}\right]e^{-bs} \end{aligned}$$

The response in the transform domain can be expressed as

$$U_{R_1}(x,s) = \frac{V_0}{T_R}\left[\frac{1}{s^2}-\frac{2q}{s^{3/2}}+\frac{2q^2}{s}-\frac{2q^2}{\sqrt{s}(\sqrt{s}+1/q)}\right]e^{-b\sqrt{s}}$$

The inverse transform of $U_{R_1}(x,s)$ can be computed by inverting each term of the above equation using the identity [13, 14]

$$\frac{F(\sqrt{s})}{\sqrt{s}} \Longleftrightarrow \frac{1}{\sqrt{\pi t}}\int_{u=0}^{\infty} e^{\frac{-u^2}{4t}} f(u)du \qquad (7)$$

Thus, the time-domain response for the first reflected diffusion is

$$\begin{aligned} u_{R_1}(x,t) &= \frac{V_0}{T_R}\Bigg[(t+\frac{b^2}{2})erfc(\frac{b}{\sqrt{4t}})-b\sqrt{\frac{t}{\pi}}e^{-\frac{b^2}{4t}} \\ &\quad -4q\sqrt{\frac{t}{\pi}}e^{-\frac{b^2}{4t}}+(2qb+2q^2)erfc\left(\frac{b}{\sqrt{4t}}\right) \\ &\quad -2q^2e^{\left(\frac{t+qb}{q^2}\right)}erfc\left(\frac{\sqrt{t}}{q}+\frac{b}{2\sqrt{t}}\right)\Bigg]U(t) \end{aligned}$$

For a finite ramp input the time-domain response is given by

$$v_{R_1}(x,t) = u_{R_1}(x,t) - u_{R_1}(x,t-T_R)$$

Second Reflection. The voltage response of the second reflected diffusion component due to the source discontinuity with $\Gamma_S(s) = \frac{Z_S-Z_0}{Z_S+Z_0} = -1$ is given by

$$V_{R_2}(x,s) = \Gamma_S\Gamma_L V_I(2h+x,s) = -\Gamma_L V_I(2h+x,s).$$

The time-domain response for infinite ramp input can be calculated from the first reflected diffusion component as

$$\begin{aligned} u_{R_2}(x,t) &= \frac{V_0}{T_R}\Bigg[-(t+\frac{b^2}{2})erfc(\frac{b}{\sqrt{4t}})+b\sqrt{\frac{t}{\pi}}e^{-\frac{b^2}{4t}} \\ &\quad +4q\sqrt{\frac{t}{\pi}}e^{-\frac{b^2}{4t}}-(2qb+2q^2)erfc\left(\frac{b}{\sqrt{4t}}\right) \\ &\quad +2q^2e^{\left(\frac{t+qb}{q^2}\right)}erfc\left(\frac{\sqrt{t}}{q}+\frac{b}{2\sqrt{t}}\right)\Bigg]U(t) \end{aligned}$$

where $b = \frac{(2h+x)}{h}\sqrt{R_hC_h}$, whence the time-domain response under finite ramp input is

$$v_{R_2}(x,t) = u_{R_2}(x,t) - u_{R_2}(x,t-T_R).$$

Third Reflection. In the transform domain, the third reflected diffusion component is

$$\begin{aligned} V_{R_3}(x,s) &= \Gamma_S{\Gamma_L}^2 V_I(4h-x,s) \\ &= \frac{-V_0(1-e^{-sT_R})}{T_R s^2}\frac{(1-q\sqrt{s})^2}{(1+q\sqrt{s})^2}e^{-\frac{(4h-x)}{h}\sqrt{sR_hC_h}} \end{aligned}$$

Considering the function

$$\begin{aligned} F_3(s) &= -\frac{V_0}{T_R s^3}\left(\frac{1-qs}{1+qs}\right)^2 e^{-bs} \\ &= \frac{V_0}{T_R}\Bigg[-\frac{1}{s^3}+\frac{4q}{s^2}-\frac{8q^2}{s}+\frac{8q^2}{(s+1/q)} \\ &\quad +\frac{4q}{(s+1/q)^2}\Bigg]e^{-bs} \end{aligned}$$

where $b = \frac{(4h-x)}{h}\sqrt{R_hC_h}$. The infinite ramp response is

$$\begin{aligned} U_{R_3}(x,s) &= \frac{-V_0}{T_R s^2}\frac{(1-q\sqrt{s})^2}{(1+q\sqrt{s})^2}e^{-b\sqrt{s}} = \frac{F_3(\sqrt{s})}{\sqrt{s}} \\ &= \frac{V_0}{T_R}\Bigg[-\frac{1}{s^2}+\frac{4q}{s^{3/2}}-\frac{8q^2}{s}+\frac{8q^2}{\sqrt{s}(\sqrt{s}+1/q)} \\ &\quad +\frac{4q}{\sqrt{s}(\sqrt{s}+1/q)^2}\Bigg]e^{-b\sqrt{s}}. \end{aligned}$$

The inverse transform of $U_{R_3}(x,s)$ can be computed by inverting each term of the above equation; the inverse transform of the first four terms can be obtained from the analysis of the **First Reflection** above. The time-domain expression for the last term can be calculated by considering the function $F_4(s) = \frac{4q}{(s+1/q)^2}e^{-bs}$ whose corresponding time-domain function is $f_4(t) = 4q(t-b)e^{-\frac{(t-b)}{q}}U(t-b)$ [14]. The time-domain response for the third reflected diffusion for an infinite ramp is

$$\begin{aligned} u_{R_3}(x,t) &= \frac{V_0}{T_R}\Bigg[-(t+\frac{b^2}{2})erfc(\frac{b}{\sqrt{4t}})+b\sqrt{\frac{t}{\pi}}e^{-\frac{b^2}{4t}} \\ &\quad +8q\sqrt{\frac{t}{\pi}}e^{-\frac{b^2}{4t}}-(4qb+8q^2)erfc\left(\frac{b}{\sqrt{4t}}\right) \\ &\quad +8q^2e^{\left(\frac{t+qb}{q^2}\right)}erfc\left(\frac{\sqrt{t}}{q}+\frac{b}{2\sqrt{t}}\right) \\ &\quad +8\sqrt{\frac{t}{\pi}}e^{\left(-(\frac{\sqrt{t}}{q}+\frac{b}{2\sqrt{t}})^2+\frac{(t+qb)}{q^2}\right)} \\ &\quad -4(2t+qb)e^{\frac{(t+qb)}{q^2}}erfc\left(\frac{\sqrt{t}}{q}+\frac{b}{2\sqrt{t}}\right)\Bigg]U(t) \end{aligned}$$

The finite ramp time-domain response is $v_{R_3}(x,t) = u_{R_3}(x,t) - u_{R_3}(x,t-T_R)$.

Fourth Reflection. Similarly, the voltage response of the fourth reflection at the source is $V_{R_4}(x,s) = {\Gamma_S}^2{\Gamma_L}^2V_I(4h+x,s) = {\Gamma_L}^2V_I(4h+x,s)$ and the time-domain response for infinite ramp input is

$$\begin{aligned} u_{R_4}(x,t) &= \frac{V_0}{T_R}\Bigg[(t+\frac{b^2}{2})erfc(\frac{b}{\sqrt{4t}})-b\sqrt{\frac{t}{\pi}}e^{-\frac{b^2}{4t}} \\ &\quad -8q\sqrt{\frac{t}{\pi}}e^{-\frac{b^2}{4t}}+(4qb+8q^2)erfc\left(\frac{b}{\sqrt{4t}}\right) \end{aligned}$$

Load factor	10% Threshold delays (R_hC_h)		50% Threshold delays (R_hC_h)	
($\frac{C_L}{C_h}$)	SPICE	Diff4	SPICE	Diff4
0.0	6.20	6.30	14.20	14.25
0.25	7.60	7.70	16.90	16.85
0.5	8.70	8.80	19.35	19.30
1.0	10.50	10.50	24.30	24.25
2.0	13.10	13.15	34.20	33.50
5.0	18.50	17.55	64.00	56.00
Load factor	**63.2% Threshold delays (R_hC_h)**		**90% Threshold delays (R_hC_h)**	
($\frac{C_L}{C_h}$)	**SPICE**	**Diff4**	**SPICE**	**Diff4**
0.0	16.50	16.50	24.75	24.70
0.25	19.75	19.70	32.00	30.00
0.5	23.30	23.30	40.00	38.90
1.0	30.50	30.60	57.50	57.75
2.0	45.10	44.70	92.00	99.00
5.0	89.50	80.00	199.00	235.00

Table 1: Comparison of delay values at the end of the interconnect line ($x = h$), between the SPICE URC model and the **Diff4** analytical expression computed using diffusion analysis up to the first four reflection components. The input rise time is assumed equal to the line time constant, i.e., $T_R = R_hC_h = 15$ psec.

$$
\begin{aligned}
&-8q^2 e^{\left(\frac{t+qb}{q^2}\right)} erfc\left(\frac{\sqrt{t}}{q}+\frac{b}{2\sqrt{t}}\right) \\
&-8\sqrt{\frac{t}{\pi}}\, e^{\left(-\left(\frac{\sqrt{t}}{q}+\frac{b}{2\sqrt{t}}\right)^2+\frac{(t+qb)}{q^2}\right)} \\
&+4(2t+qb)e^{\frac{(t+qb)}{q^2}} erfc\left(\frac{\sqrt{t}}{q}+\frac{b}{2\sqrt{t}}\right)\Bigg] U(t)
\end{aligned}
$$

where $b = \frac{(4h+x)}{h}\sqrt{R_hC_h}$; the response for the finite ramp input is $v_{R_4}(x,t) = u_{R_4}(x,t) - u_{R_4}(x,t-T_R)$. If we approximate the total response for infinite ramp input by considering only up to these first four reflections,

$$
\begin{aligned}
u_{Tot}(x,t) &\approx u_I(x,t)+u_{R_1}(x,t)+u_{R_2}(x,t) \\
&\quad + u_{R_3}(x,t)+u_{R_4}(x,t)
\end{aligned} \tag{8}
$$

and the total response for the finite ramp input is

$$
\begin{aligned}
v_{Tot}(x,t) &= u_{tot}(x,t) - u_{Tot}(x,t-T_R) \\
&\approx \left[v_I(x,t)+v_{R_1}(x,t)+v_{R_2}(x,t)\right. \\
&\quad \left.+ v_{R_3}(x,t)+v_{R_4}(x,t)\right]
\end{aligned} \tag{9}
$$

We call the approximation of Equation (9) the **Diff4** model. Table 1 compares **Diff4** delay estimates at different threshold values for a wide range of capacitive loads, versus the SPICE URC (Uniform Distributed *RC*) model. The delay estimates using our new diffusion equation approach are very close to the SPICE-computed delays, even though only four reflections are considered. Figure 7 gives a comparison of the voltage response between SPICE and the **Diff4** model for the case of load factor $\frac{C_L}{C_h} = 1.0$.

6 Generalization of the Reflected Component Computation

While the previous section gave analytical expressions for the first four reflection components, we now discuss methods to compute higher-order components of the infinite ramp response; from these, the components of the finite ramp response easily follow. In general, the (infinite or finite) ramp response in the transform domain is a function of reflection coefficients and the incident voltage. From Equation (5) the $2n^{th}$ reflected component at the source is given by $U_{R_{2n}}(x,s) = \Gamma_S{}^n \Gamma_L{}^n\, U_I(2nh+x,s)$, and the $(2n-1)^{th}$ reflected component at the load is similarly given by $U_{R_{2n-1}}(x,s) = \Gamma_S{}^{n-1}\Gamma_L{}^n U_I(2nh-x,s)$. The time-domain response for each reflected diffusion component can be computed using the above method (Section 5) for obtaining an exact analytical expression. To generalize the computation of each time-domain reflected diffusion component we may apply two different techniques: (i) a numerical integration approach, and (ii) recursive error function evaluation.

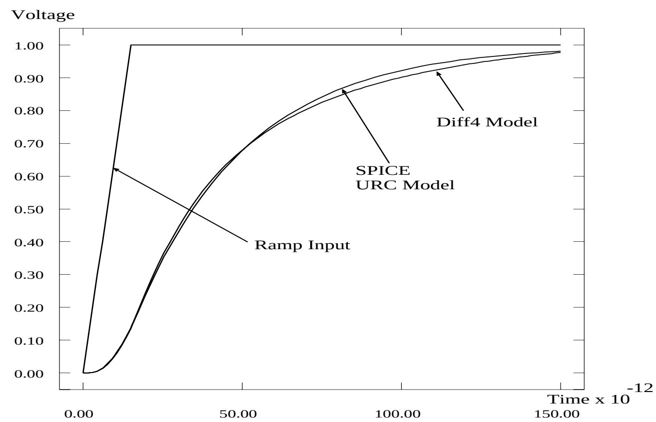

Figure 7: Response for a finite *RC* line with capacitive load under finite ramp input, calculated using both the SPICE URC model and the **Diff4** model. The rise time of the input is $T_R = R_hC_h$ and the load factor is $\frac{C_L}{C_h} = 1.0$. The line parameters are same as in Figure 5.

6.1 A Numerical Integration Approach

Here and in the next subsection, we consider the time-domain expression for the $2n^{th}$ source reflection component (the $(2n-1)^{th}$ load reflection component is analogous). Assuming resistive source and capacitive load impedances, the reflection coefficients can be represented as $\Gamma_S(s) = -\frac{(1-p\sqrt{s})}{(1+p\sqrt{s})}$ $\quad \Gamma_L(s) = \frac{(1-q\sqrt{s})}{(1+q\sqrt{s})}$ where $p = \frac{R_S}{R_h}\sqrt{R_hC_h}$ and $q = \frac{C_L}{C_h}\sqrt{R_hC_h}$. Substituting for the reflection coefficients and incident voltage and using $b = \frac{(2nh+x)}{h}\sqrt{R_hC_h}$, the infinite ramp response in the transform domain is given by

$$
\begin{aligned}
U_{R_{2n}}(x,s) &= \Gamma_L^n(s)\Gamma_S^n(s)V_I(2nh+x,s) = \Gamma_L^n(s)\Gamma_S^n(s)\frac{V_0 e^{-b\sqrt{s}}}{T_R s^2} \\
&= (-1)^n \frac{V_0}{T_R s^2}\frac{(1-p\sqrt{s})^n}{(1+p\sqrt{s})^{n+1}}\frac{(1-q\sqrt{s})^n}{(1+q\sqrt{s})^n} e^{-b\sqrt{s}}
\end{aligned}
$$

The response in the time domain for this reflected component can be evaluated by expressing $U_{R_{2n}}$ in the form $\frac{F(\sqrt{s})}{\sqrt{s}}$ and applying the identity in Equation (7) [14]. To evaluate the integral in the identity we need the time-domain expression $f(t)$ of the function. Hence, we first apply partial fraction expansion to the function as

$$
\begin{aligned}
F(s) &= (-1)^n \frac{V_0}{T_R s^3}\frac{(1-ps)^n}{(1+ps)^{n+1}}\frac{(1-qs)^n}{(1+qs)^n} e^{-bs} \\
&= \frac{V_0(-1)^n}{pT_R}\left[\frac{D_1}{s}+\frac{D_2}{s^2}+\frac{D_3}{s^3}\right. \\
&\quad + \frac{A_1}{(s+1/p)}+\ldots+\frac{A_{n+1}}{(s+1/p)^{n+1}} \\
&\quad \left.+ \frac{B_1}{(s+1/q)}+\ldots+\frac{B_n}{(s+1/q)^n}\right] e^{-bs}
\end{aligned}
$$

where A_i, B_i, and D_i are the coefficients corresponding to each pole of the function. The inverse transform for this function is

$$f(t) = \frac{V_0(-1)^n}{pT_R}\left[D_1 + D_2(t-b) + D_3\frac{(t-b)^2}{2} + A_1 e^{(t-b)/p} + \ldots + \frac{A_{n+1}}{n!}(t-b)^n e^{(t-b)/p} + B_1 e^{(t-b)/q} + \ldots + \frac{B_n(t-b)^{n-1}}{(n-1)!}e^{(t-b)/q}\right]U(t-b)$$

and the time-domain response for the reflection component can be calculated by numerical integration as $u_{R_{2n}}(x,t) = \frac{1}{\sqrt{\pi t}}\int_{x=0}^{\infty} e^{\frac{-x^2}{4t}} f(x)dx$.

6.2 Recursive Error Function Evaluation

Instead of calculating the inverse transform of the function $F(s)$ and using numerical integration, we may rewrite the reflected component in the transform domain in the form $\frac{F(\sqrt{s})}{\sqrt{s}}$ and then calculate the inverse transform, i.e.,

$$U_{R_{2n}}(x,s) = \frac{F(\sqrt{s})}{\sqrt{s}} = \frac{V_0(-1)^n}{pT_R}\left[\frac{D_1}{s} + \frac{D_2}{s^{3/2}} + \frac{D_3}{s^2} + \frac{A_1}{\sqrt{s}(\sqrt{s}+1/p)} + \ldots + \frac{A_{n+1}}{\sqrt{s}(\sqrt{s}+1/p)^{n+1}} + \frac{B_1}{\sqrt{s}(\sqrt{s}+1/q)} + \ldots + \frac{B_n}{\sqrt{s}(\sqrt{s}+1/q)^n}\right]e^{-b\sqrt{s}}$$

The inverse transform for $U_{R_{2n}}(x,s)$ can now be obtained by taking inverse transforms separately for each term in the above expression. The time-domain expression for the $2n^{th}$ reflection component can be obtained in the form of recursive error functions as[14]

$$u_{R_{2n}}(x,t) = \frac{V_0(-1)^n}{pT_R}\left[D_1 erfc\left(\frac{b}{\sqrt{4t}}\right) + 2D_2\sqrt{\frac{t}{\pi}}e^{-\frac{b^2}{4t}} - bD_2 erfc\left(\frac{b}{\sqrt{4t}}\right) + D_3(t+\frac{b^2}{2})erfc(\frac{b}{\sqrt{4t}}) - bD_3\sqrt{\frac{t}{\pi}}e^{-\frac{b^2}{4t}} + \sum_{k=1}^{n+1} A_k(4t)^{\frac{(n-1)}{2}} e^{(\frac{t}{p^2}-\frac{b}{p})} erfc_{(n-1)}\left(\frac{b-\frac{2t}{p}}{\sqrt{4t}}\right) \sum_{k=1}^{n} B_k(4t)^{\frac{(n-1)}{2}} e^{(\frac{t}{q^2}-\frac{b}{q})} erfc_{(n-1)}\left(\frac{b-\frac{2t}{q}}{\sqrt{4t}}\right)\right] \quad (10)$$

which can be evaluated using the recursive expression for the error function [5, 14], i.e.,

$$erfc_{(n)}(z) = -\frac{z}{n}erfc_{(n-1)}(z) + \frac{1}{2n}erfc_{(n-2)}(z) = \frac{2}{\sqrt{\pi}}\int_{t=z}^{\infty}\frac{(t-z)^n}{n!}e^{-t^2}dt .$$

Thus, the time-domain response for the finite ramp input can be obtained as $v_{R_{2n}}(x,t) = u_{R_{2n}}(x,t) - u_{R_{2n}}(x,t-T_R)$.

7 Conclusions

We have analyzed finite distributed *RC* lines under ramp input via a new technique based on solving the diffusion equation and using reflected diffusion components to account for reflections at the source and load end of the line. Our general and, in the limit, *exact* approach computes the time-domain response for finite *RC* lines under ramp input by summing distinct diffusions starting at either end of the line. We then derived the time-domain voltage response for various configurations of the *RC* line. To the best of our knowledge, these results are completely new; there is no previous literature on this subject. Delay estimates using our new approach (the **Diff4** model incorporating up to the first four reflection components) are very close to SPICE-computed (URC model) delays. Finally, we present two methodologies, including a general recursive equation, for computing the higher-order diffusion components due to reflections at either the source or load end. Ongoing work extends this approach to response computations in arbitrary interconnection trees by modeling both reflection and transmission coefficients at discontinuities, e.g., we might derive the input ramp for each interconnect from the response at the end of the previous (upstream) interconnect.

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