Exercises (strongly recommended, do not turn in)

1. (DPV textbook, Problem 7.11.) Find the optimal solution to the following linear program.

\[
\begin{align*}
\text{max} & \quad x + y \\
2x + y & \leq 3 \\
x + 3y & \leq 5 \\
x, y & \geq 0
\end{align*}
\]

2. (DPV textbook, Problem 7.21.) An edge of a flow network is called critical if decreasing the capacity of this edge results in a decrease in the maximum flow. Give an efficient algorithm that finds a critical edge in a network.

3. (DPV textbook, Problem 7.23.) The vertex cover of an undirected graph \(G = (V,E)\) is a subset of the vertices which touches every edge- that is, a subset \(S \subseteq V\) such that for each edge \(u, v \in E\), one or both of \(u, v\) are in \(S\).

Show that the problem of finding the minimum vertex cover in a bipartite graph reduces to maximum flow. (Hint: Can you relate this to the minimum cut in a related network?)

Problems (must be written up and turned in)

1. DPV 6.2: You are going on a long trip. You start on the road at mile post 0. Along the way there are \(n\) hotels, at mile posts \(a_1 < a_2 < \ldots < a_n\), where each \(a_i\) is measured from the starting point. The only places you are allowed to stop are at these hotels, but you can choose which of the hotels you stop at. You must stop at the final hotel (at distance \(a_n\)), which is your destination.

You’d ideally like to travel 200 miles a day, but this may not be possible (depending on the spacing of the hotels). If you travel \(x\) miles during a day, the penalty for that day is \((200 - x)^2\). You want to plan your trip so as to minimize the total penalty- that is, the sum, over all travel days, of the daily penalties. Give an efficient algorithm that determines the optimal sequence of hotels at which to stop.

2. DPV 6.13: Consider the following game. A “dealer” produces a sequence \(s_1 \ldots s_n\) of “cards”, face up, where each card \(s_i\) has a value \(v_i\). Then two players take turns picking a card from the sequence, but can only pick the first or the last card of the (remaining) sequence. The goal is to collect cards of largest total value. (For example, you can think of the cards as bills of different denominations.) Assume \(n\) is even.

(a) Show a sequence of cards such that it is not optimal for the first player to start by picking up the available card of larger value. That is, the natural greedy strategy is suboptimal.

(b) Give an \(O(n^2)\) algorithm to compute an optimal strategy for the first player. Given the initial sequence, your algorithm should precompute in \(O(n^2)\) time some information, and then the first player should be able to make each move optimally in \(O(1)\) time by looking up the precomputed information.

3. Every morning you purchase the daily donut from the exotic bakery Dynamic Pastries. You have a schedule over a period of \(n\) days for which the donut on day \(i\) costs \(p_i\). They also offer a special where, on any given day, you can prepay for the next 13 days for a fixed price \(C\). You don’t want to have any prepaid days left over at the end of the \(n\) days, so you can’t purchase the special if there are less than 13 days remaining. Given a set of \(n\) prices \(p_1, \ldots, p_n\), the special cost \(C\), and your insane donut addiction, you want to figure out the minimum amount of money that you need to cover the \(n\) days.
(a) You first try a greedy approach. For any day $i$, you compare special price $C$ versus the cost of purchasing individually for the next 13 days ($p_i + p_{i+1} + \cdots + p_{i+12}$). If the special is cheaper, you buy it; otherwise you buy the donut individually for that day. Give an example where this strategy leads you to spending more than needed.

(b) Give an efficient dynamic programming algorithm to find the optimal solution. Be sure to provide a recursive formulation and analyze the time complexity.

4. DPV 7.5: The Canine Products company offers two dog foods, Frisky Pup and Husky Hound, that are made from a blend of cereal and meat. A package of Frisky Pup requires 1 pound of cereal and 1.5 pounds of meat, and sells for $7. A package of Husky Hound uses 2 pounds of cereal and 1 pound of meat, and sells for $6. Raw cereal costs $1 per pound and raw meat costs $2 per pound. It also costs $1.40 to package the Frisky Pup and $0.60 to package the Husky Hound. A total of 240,000 pounds of cereal and 180,000 pounds of meat are available each month. The only production bottleneck is that the factory can only package 110,000 bags of Frisky Pup per month. Needless to say, management would like to maximize profit.

(a) Formulate the problem as a linear program in two variables.

(b) Graph the feasible region, give the coordinates of every vertex, and circle the vertex maximizing profit. What is the maximum profit possible?

5. DPV 7.18: There are many common variations of the maximum flow problem. Here are four of them.

(a) There are many sources and many sinks, and we wish to maximize the total flow from all sources to all sinks.

(b) Each vertex also has a capacity on the maximum flow that can enter it.

(c) Each edge has not only a capacity, but also a lower bound on the flow it must carry.

(d) The outgoing flow from each node $u$ is not the same as the incoming flow, but is smaller by a factor of $(1 - \epsilon_u)$, where $\epsilon_u$ is a loss coefficient associated with node $u$.

Each of these can be solved efficiently. Show this by reducing (a) and (b) to the original max-flow problem, and reducing (c) and (d) to linear programming.