Conditional Probability. Expected Value.

CSE21 Winter 2017, Day 22 (B00), Day 14-15 (A00)

March 8, 2017

http://vlsicad.ucsd.edu/courses/cse21-w17
A random variable assigns a real number to each possible outcome of an experiment.
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The distribution of a random variable $X$ is the function

$$r \rightarrow P(X = r)$$

$\mathbb{R} \rightarrow [0, 1]$
A **random variable** assigns a real number to each possible outcome of an experiment.

The distribution of a random variable $X$ is the function

$$r \mapsto P(X = r)$$

The **expectation** (average, expected value) of random variable $X$ on sample space $S$ is

$$E(X) = \sum_{s \in S} P(s)X(s) = \sum_{r \in X(S)} P(X = r)r$$

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*Rosen p. 460,478*
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$$= \sum_{r \in X(S)} P(X = r)r$$

Calculate the expected number of boys in a family with two children.

A. 0  
B. 1  
C. 1.5  
D. 2
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$$E(X) = \sum_{s \in S} P(s)X(s)$$

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Calculate the expected number of boys in a family with three children.

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B. 1
C. 1.5
D. 2
The expectation (average, expected value) of random variable $X$ on sample space $S$ is

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Calculate the expected number of boys in a family with three children.

A. 0  
B. 1  
C. 1.5  
D. 2  

The expected value might not be a possible value of the random variable… like 1.5 boys!
The **expectation** (average, expected value) of random variable $X$ on sample space $S$ is

$$E(X) = \sum_{s \in S} P(s) X(s)$$

$$= \sum_{r \in X(S)} P(X = r) r$$

Calculate the expected sum of two 6-sided dice.

A. 6  
B. 7  
C. 8  
D. 9  
E. None of the above.
$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

$$E(x) = P(x=2)2 + P(x=3)3 + P(x=4)4 + P(x=5)5 + \ldots$$
Properties of Expectation

- $E(X)$ may not be an actually possible value of $X$.

- But $m \leq E(X) \leq M$, where
  - $m$ is the minimum possible value of $X$ and
  - $M$ is the maximum possible value of $X$.
Useful trick 1: Case analysis

The expectation can be computed by conditioning on an event and its complement.

**Theorem:** For any random variable $X$ and event $A$,

$$E(X) = P(A)E(X | A) + P(A^c)E(X | A^c)$$

where $A^c$ is the complement of $A$. 

*Rosen p. 460,478*
Useful trick 1: Case analysis

**Example**: If $X$ is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of $X$?

e.g. $X(\text{HHT}) = 1$
$X(\text{HHH}) = 2.$

$X(\text{THH}) = 0$
Useful trick 1: Case analysis

**Example:** If $X$ is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of $X$?

**Solution:**

$$E(X) = \sum_{s \in S} P(s)X(s)$$

*Directly from definition*

$$= \sum_{r \in X(S)} P(X = r)r$$

For each of eight possible outcomes, find probability and value of $X$:

HHH ($P(\text{HHH}) = 1/8$, $X(\text{HHH}) = 2$), HHT, HTH, HTT, THH, THT, TTH, TTT etc.
Useful trick 1: Case analysis

Example: If $X$ is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of $X$?

Solution:

*Using conditional expectation*

Let $A$ be the event "The middle flip is H".

Which subset of $S$ is $A$?

A. $\{ \text{HHH} \}$  
B. $\{ \text{THT} \}$  
C. $\{ \text{HHT, THH} \}$  
D. $\{ \text{HHH, HHT, THH, THT} \}$  
E. None of the above.
Example: If $X$ is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of $X$?

Solution:

*Using conditional expectation*

Let $A$ be the event "The middle flip is H".

$$E(X) = P(A) E(X | A) + P(A^c) E(X | A^c)$$

$$E(X) = \frac{1}{2} E(X | A) + \frac{1}{2} E(X | A^c)$$
Useful trick 1: Case analysis

**Example**: If $X$ is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of $X$?

**Solution**:

*Using conditional expectation*

Let $A$ be the event "The middle flip is H". 

\[ P(A) = \frac{1}{2} \; , \; P(A^c) = \frac{1}{2} \]

\[ E(X) = P(A) \, E(X \mid A) + P(A^c) \, E(X \mid A^c) \]
Useful trick 1: Case analysis

Example: If $X$ is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of $X$?

Solution:

*Using conditional expectation*

Let $A$ be the event "The middle flip is H". $P(A) = 1/2$, $P(A^c) = 1/2$

$$E(X) = P(A) E(X | A) + P(A^c) E(X | A^c)$$

$E(X | A^c)$: If middle flip isn't H, there can't be any pairs of consecutive Hs
Useful trick 1: Case analysis

**Example**: If $X$ is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of $X$?

**Solution**: 

*Using conditional expectation*

Let $A$ be the event "The middle flip is H". 

\[ P(A) = 1/2 \text{, } P(A^c) = 1/2 \]

\[ E(X) = P(A) E(X | A) + P(A^c) E(X | A^c) \]

$E(X | A^c)$: If middle flip isn't H, there can't be *any* pairs of consecutive Hs

$E(X | A)$: If middle flip is H, # pairs of consecutive Hs = # Hs in first & last flips
Useful trick 1: Case analysis

**Example**: If $X$ is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of $X$?

**Solution**:

*Using conditional expectation*

Let $A$ be the event "The middle flip is H". 

- $P(A) = 1/2$ , $P(A^c) = 1/2$

$$E(X) = P(A) E(X | A) + P(A^c) E(X | A^c)$$

- $E(X | A^c) = 0$

- $E(X | A) = \frac{1}{4} * 0 + \frac{1}{2} * 1 + \frac{1}{4} * 2 = 1$
Useful trick 1: Case analysis

Example: If $X$ is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of $X$?

Solution:

*Using conditional expectation*

Let $A$ be the event "The middle flip is H".  

$$P(A) = 1/2, \quad P(A^c) = 1/2$$

$$E(X) = P(A) E(X | A) + P(A^c) E(X | A^c) = \frac{1}{2} (1) + \frac{1}{2} (0) = 1/2$$

$$E(X | A^c) = 0$$

$$E(X | A) = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$
Useful trick 1: Case analysis

Examples: Ending condition
• Each time I play solitaire I have a probability $p$ of winning. I play until I win a game.
• Each time a child is born, it has probability $p$ of being left-handed. I keep having kids until I have a left-handed one.

Let $X$ be the number of games OR number of kids until ending condition is met.

What's $E(X)$?

A. 1.
B. Some big number that depends on $p$.
C. $1/p$.
D. None of the above.
Useful trick 1: Case analysis

Ending condition

Let $X$ be the number of games OR number of kids until ending condition is met.

Solution:

*Directly from definition*

Need to compute the sum of all possible $P(X = i) i$. 

Useful trick 1: Case analysis

Ending condition

Let $X$ be the number of games OR number of kids until ending condition is met.

Solution:

*Directly from definition*

Need to compute the sum of all possible $P(X = i)$.

$P(X = i) = \text{Probability that don't stop the first } i-1 \text{ times and do stop at the } i^{th} \text{ time}$

$= (1-p)^{i-1} \cdot p$
Ending condition

Let $X$ be the number of games OR number of kids until ending condition is met.

Solution:

Directly from definition

Need to compute the sum of all possible $P(X = i) i$.

$P(X = i) = \text{Probability that don't stop the first } i-1 \text{ times and do stop at the } i^{th} \text{ time} = (1-p)^{i-1} p$

$$E(x) = \sum_{i=1}^{\infty} i(1-p)^{i-1}p$$
Ending condition

Let $X$ be the number of games OR number of kids until ending condition is met.

Solution:

Using conditional expectation

$$E(X) = P(A) E(X \mid A) + P(A^c) E(X \mid A^c)$$

$A$ is "success at first try"
Useful trick 1: Case analysis

Ending condition

Let $X$ be the number of games OR number of kids until ending condition is met.

Solution:

*Using conditional expectation*

Let $A$ be the event "success at first try".

$$E(X) = P(A) E(X | A) + P(A^c) E(X | A^c)$$

$$P(A) = p$$

$$P(A^c) = 1 - p$$
Useful trick 1: Case analysis

Ending condition

Let $X$ be the number of games OR number of kids until ending condition is met.

Solution:

Using conditional expectation

Let $A$ be the event "success at first try".

$$E(X) = P(A) E(X | A) + P(A^c) E(X | A^c)$$

$$P(A) = p$$
$$P(A^c) = 1-p$$

$$E(X | A) = 1$$
Useful trick 1: Case analysis

Ending condition

Let X be the number of games OR number of kids until ending condition is met.

Solution:

Using conditional expectation

Let A be the event "success at first try".

\[ E(X) = P(A) E(X | A) + P(A^c) E(X | A^c) \]

\[ P(A) = p \quad P(A^c) = 1 - p \]

\[ E(X|A) = 1 \quad \text{because stop after first try} \]

\[ E(X | A^c) = 1 + E(X) \]
Useful trick 1: Case analysis

Ending condition

Let $X$ be the number of games OR number of kids until ending condition is met.

Solution:

*Using conditional expectation*

Let $A$ be the event "success at first try".

$$E(X) = P(A) E(X | A) + P(A^c) E(X | A^c)$$

$P(A) = p$

$P(A^c) = 1-p$

$E(X|A) = 1$

$E(X|A^c) = 1 + E(X)$ because tried once and then at same situation from start
Useful trick 1: Case analysis

Ending condition

Let $X$ be the number of games OR number of kids until ending condition is met.

Solution:

*Using conditional expectation*

Let $A$ be the event "success at first try".

$$E(X) = P(A) E(X | A) + P(A^c) E(X | A^c)$$

$$P(A) = p$$

$$P(A^c) = 1 - p$$

$$E(X|A) = 1$$

$$E(X|A^c) = 1 + E(X)$$

$$E(X) = p(1) + (1-p)(1 + E(X))$$
Useful trick 1: Case analysis

Ending condition

Let $X$ be the number of games OR number of kids until ending condition is met.

Solution:

*Using conditional expectation*

Let $A$ be the event "success at first try".

Solving for $E(X)$ gives:
**Theorem:** If $X_i$ are random variables on $S$ and if $a$ and $b$ are real numbers then

$$E(X_1 + \ldots + X_n) = E(X_1) + \ldots + E(X_n)$$

and

$$E(aX + b) = aE(x) + b.$$
Useful trick 2: Linearity of expectation

**Example:** Expected number of *consecutive heads* when we flip a fair coin n times?

A. 1.
B. 2.
C. n.
D. None of the above.
E. ??
Example: Expected number of consecutive heads when we flip a fair coin n times?

Solution: Define $X_i = 1$ if both the $i^{th}$ and $(i+1)^{st}$ flips are H; $X_i = 0$ otherwise.

Looking for $E(X)$ where

$$X = \sum_{i=1}^{n-1} X_i$$

For each $i$, what is $E(X_i)$?

A. 0.
B. $\frac{1}{4}$.
C. $\frac{1}{2}$.
D. 1.
E. It depends on the value of $i$. 
Useful trick 2: Linearity of expectation

Example: Expected number of consecutive heads when we flip a fair coin n times?

Solution: Define $X_i = 1$ if both the $i^{th}$ and $(i+1)^{st}$ flips are H; $X_i = 0$ otherwise.

Looking for $E(X)$ where 

$$X = \sum_{i=1}^{n-1} X_i$$

$$E(X) = \sum_{i=1}^{n-1} E(X_i) = \sum_{i=1}^{n-1} \frac{1}{4} = \frac{n - 1}{4}$$
Useful trick 2: Linearity of expectation

Example: Expected number of **consecutive heads** when we flip a fair coin \( n \) times?

Solution: Define \( X_i = 1 \) if both the \( i \)th and \( (i+1) \)st flips are H; \( X_i = 0 \) otherwise.

Looking for \( E(X) \) where

\[
X = \sum_{i=1}^{n-1} X_i
\]

\[
E(X) = \sum_{i=1}^{n-1} E(X_i) = \sum_{i=1}^{n-1} \frac{1}{4} = \frac{n - 1}{4}
\]

**Indicator variables:** 1 if pattern occurs, 0 otherwise.
Expectation does not in general commute with other functions.

\[ E( f(X) ) \neq f( E(X) ) \]

For example, let \( X \) be random variable with \( P(X = 0) = \frac{1}{2} \), \( P(X = 1) = \frac{1}{2} \)

What's \( E(X) \)?

What's \( E(X^2) \)?

What's \( (E(X))^2 \)?
Other functions?

Expectation does not in general commute with other functions.

\[ E \left( f(X) \right) \neq f \left( E(X) \right) \]

For example, let \( X \) be random variable with \( P(X = 0) = \frac{1}{2} \), \( P(X = 1) = \frac{1}{2} \)

What's \( E(X) \)? \( \frac{1}{2} \times 0 + \frac{1}{2} \times 1 = \frac{1}{2} \)

What's \( E(X^2) \)? \( \frac{1}{2} \times 0^2 + \frac{1}{2} \times 1^2 = \frac{1}{2} \)

What's \( (E(X))^2 \)? \( \left( \frac{1}{2} \right)^2 = \frac{1}{4} \)