Graphs and Puzzles. Eulerian and Hamiltonian Tours.

CSE21 Winter 2017, Day 11 (B00), Day 7 (A00)

February 3, 2017

http://vlsicad.ucsd.edu/courses/cse21-w17
Exam Announcements

Exam on Monday
Covers through Day 9
(no graphs)

- Bring student ID.
- One handwritten note sheet
  (8.5” x 11”, both sides).
- Look up seat assignment.
- No calculators.
- No blue books.

Review Session
covering the Practice Midterm

Tomorrow 1-3pm
Sunday noon-2pm

Seating Chart on Website

Good luck!
Representing directed graphs

**Adjacency list** (list of lists):
for each vertex v, associate list of all **neighbors** of v.

Rosen p. 668
Neighbors

The **neighbors** of a vertex $v$ are all the vertices $w$ for which there is an edge whose endpoints are $v,w$.

If two vertices are neighbors then they are called **adjacent** to one another.

Rosen p. 651
The degree of a vertex in an undirected graph is the total number of edges incident with it, except that a loop contributes twice.

What's the maximum degree of a vertex in this graph?

A. 0.
B. 1
C. 2
D. 3
E. None of the above.
What's the degree of vertex 5?
A. 5
B. 3
C. 2
D. 1
E. None of the above.
What's the degree of vertex 0?
A. 5
B. 3
C. 2
D. 1
E. None of the above.
If there are \( n \) people in a room, and each shakes hands with \( d \) people, how many handshakes take place?

A. \( n \)
B. \( d \)
C. \( nd \)
D. \( (nd)/2 \)
E. None of the above.
Handshakes

If there are \( n \) people in a room, and each shakes hands with \( d \) people, how many handshakes take place?

A. \( n \)
B. \( d \)
C. \( nd \)
D. \( \frac{nd}{2} \)
E. None of the above.

Don't double-count each handshake!
If a simple graph has \textbf{n vertices} and each vertex has \textbf{degree d}, how many edges are there?

\[ 2 |E| = n \times d \]
Handshakes “in” graphs

Undirected

If any graph has \textbf{n vertices}, then

\[ 2 |E| = \sum_{v \in V} \deg(v). \]
Handshakes "in" graphs

If any graph has \( n \) vertices, then

\[
2 |E| = \text{sum of degrees of all vertices}
\]

What can we conclude?
A. Every degree in the graph is even.
B. The number of edges is even.
C. The number of vertices with odd degree is even.
D. The number self loops is even.
E. None of the above.
Recall: What is a graph?

A (directed) graph $G$ is

- A nonempty set of vertices $V$, also called nodes

and

- A set of edges $E$, each pointing from one (tail) vertex to another (head) vertex. (A directed edge is denoted with an “arrow”)
Tartaglia's Pouring Problem

You can pour from one cup to another until the first is empty or the second is full.

Large cup: contains 8 ounces, can hold more.  
Medium cup: is empty, has 5 ounce capacity.  
Small cup: is empty, has 3 ounce capacity

Can we divide the coffee in half? How?
Tartaglia's Pouring Problem

You can pour from one cup to another until the first is empty or the second is full.

Can we divide the coffee in half? How?

Hint: configurations $(l,m,s)$ code # ounces in each cup

Large cup: contains 8 ounces, can hold more.
Medium cup: is empty, has 5 ounce capacity.
Small cup: is empty, has 3 ounce capacity
Tartaglia's Pouring Problem
Rephrasing the problem:
Looking for path from (8,0,0) to (4,4,0)
Path

Sequence

\((v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)\)

describes a route through the graph from

**start vertex** \(v_0\)

**end vertex** \(v_k\)

Order matters!

Such that each edge \(e_i = (v_{i-1}, v_i)\)

trivial path: every vertex has a path to itself.
Rephrasing the problem:

(1) Is there a path from \((8,0,0)\) to \((4,4,0)\) ?
(2) If so, what's the best path?

"Best" means "shortest length"
What's the shortest length of a path from (8,0,0) to (4,4,0) ?

A. 7  
B. 8  
C. 14  
D. 16  
E. None of the above.
Algorithmic questions related to paths

**Reachability**: // “decision”
Does there exist a path from vertex v to vertex w?

**Path**: // “construction”
Find a path from vertex v to vertex w.

**Distance**: // “optimization”
What’s the length of the shortest path from vertex v to vertex w?
Is there a path that crosses each bridge (exactly) once?

A. Yes
B. No

Rosen p. 693
Observe: exact location on the north side doesn't matter because must come & go via a bridge. Can represent each bridge as an edge.
Seven Bridges of Konigsberg

Is there a path where each edge occurs exactly once? \textbf{Euler (or, Eulerian) tour}

\textbf{Eulerian path}

Rosen p. 693
Seven Bridges of Konigsberg redux

Which of these figures can you draw without lifting your pencil off the paper and without retracing any edge of the figure?

A.  
B.  
C.  
D.
Algorithmic questions related to Euler tours

**Existence:**
Does the given graph $G$ contain an Euler tour?

**Path:**
Find an Euler tour for the given graph $G$, if possible.

*Turns out there are great algorithms for each of these … next!*
A Hamiltonian tour is a path where each vertex occurs exactly once.

Existence:
Does the given graph $G$ contain a Hamiltonian tour?

Path:
Find a Hamiltonian tour for the given graph $G$, if possible.

These questions turn out to be intractable!!!
Actually, it is not known how to determine in any reasonable amount of time whether a graph G has a Hamiltonian Tour, or how to find one.

**An Opportunity:**
You can earn $1,000,000 if you can give an algorithm that finds a Hamiltonian Tour (if one exists) in an arbitrary graph on n vertices that takes time $O(n^k)$ for some constant k.
A light bulb is connected to 3 switches in such a way that it lights up only when all the switches are in the proper position. But you don't know what the proper position of each switch is!

All the switches are down. How many times must you press a switch if you want to guarantee that the light bulb will turn on?

A. 4
B. 7
C. 8
D. 16
Light Switches

Configuration 1 if switch is UP, 0 if DOWN
Connect configuration if off by one switch.

Rephrasing the problem:

Looking for Hamiltonian tour through graph. (starting at the “000” vertex)
Puzzle / Problem

Model as a graph
Choose vertex set & edge set … sometimes many possible options

Use graph algorithms to solve puzzle / problem
Problem

Given collection of short DNA strings.

Find longer DNA string that includes them all.

Many possible formulations as a graph problem. Successful solution was a big step in Human Genome Project.
Given collection of short DNA strings.
S = { ATG, AGG, TGC, TCC, GTC, GGT, GCA, CAG }
Find longer DNA string that includes them all.

Vertex set:
Edge set:
Given collection of short DNA strings.
S = \{ ATG, AGG, TGC, TCC, GTC, GGT, GCA, CAG \}
Find longer DNA string that includes them all.

**Vertex set**: S, i.e. a vertex for each short DNA string

**Edge set**: edge from v to w if the first two letters of w equal the last two letters of v
Given collection of short DNA strings. 
\( S = \{ \text{ATG, AGG, TGC, TCC, GTC, GGT, GCA, CAG} \} \)
Find longer DNA string that includes them all.

**Vertex set**: \( S \), i.e. a vertex for each short DNA string

**Edge set**: edge from \( v \) to \( w \) if the **first two** letters of \( w \) equal the **last two** letters of \( v \)

What's a Hamiltonian tour?
Given collection of short DNA strings.
$S = \{ \text{ATG, AGG, TGC, TCC, GTC, GGT, GCA, CAG} \}$
Find longer DNA string that includes them all.

**Vertex set:**
**Edge set:**
Given collection of short DNA strings.
S = { ATG, AGG, TGC, TCC, GTC, GGT, GCA, CAG }
Find longer DNA string that includes them all.

**Vertex set:** All length-two strings that appear in a word in S
**Edge set:** edge from ab to bc if abc is in S.

What's an Eulerian tour?
Algorithmic questions related to tours

An **Eulerian tour** is a **path** where each **edge** occurs exactly once.
- **Eulerian cycle**: Eulerian tour that starts and ends at same vertex.

A **Hamiltonian tour** is a **path** where each **vertex** occurs exactly once.
- **Hamiltonian cycle**: Ham. tour that starts and ends at same vertex.

Which is true of the graph to the right?
A. There's only one Eulerian cycle in this graph.
B. There's no Hamiltonian tour in this graph.
C. Each Hamiltonian tour is an Eulerian cycle.
D. None of the above.
E. More than one of the above.
How many paths are there between vertex A and vertex B?

A. None.
B. Exactly one.
C. Exactly two.
D. More than two.
E. None of the above.
How many paths are there between vertex A and vertex G?

A. None.
B. Exactly one.
C. Exactly two.
D. More than two.
E. None of the above.
How many paths are there between vertex A and vertex I?

A. None.
B. Exactly one.
C. Exactly two.
D. More than two.
E. None of the above.
An undirected graph $G$ is **connected** if *for any* ordered pair of vertices $(v,w)$ there is a path from $v$ to $w$. 

**Connected**

**Not connected**
An undirected graph $G$ is **connected** if *for any* ordered pair of vertices $(v,w)$ there is a path from $v$ to $w$.

An undirected graph $G$ is **disconnected** if

A. For any ordered pair of vertices $(v,w)$ there is no path from $v$ to $w$.  
B. There is an ordered pair of vertices $(v,w)$ with a path from $v$ to $w$.  
C. There is an ordered pair of vertices $(v,w)$ with no path from $v$ to $w$.  
D. For every ordered pair of vertices $(v,w)$ there is a path from $v$ to $w$.  
E. None of the above.
Disconnected graphs can be broken up into pieces where each is connected.

Each (maximal) connected piece of the graph is a **connected component**.

“maximal” means that if you make it any larger, it will lose the given (in this case, “connected”) property.
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