1. **Recursively Defined Trees.** Let $T_1$ be the rooted tree consisting of a single root vertex. Let $T_2 = T_1$. For $n \geq 3$, let $T_n$ be defined as a rooted tree with three subtrees, with the left and right subtrees $T_{n-2}$, and the middle subtree $T_{n-1}$.

(a) Draw the tree $T_5$.
(b) Let $L(n)$ be the number of leaves in $T_n$. Find a recurrence that $L(n)$ satisfies.
(c) Let $I(n)$ be the number of internal nodes in $T_n$. Find a recurrence that $I(n)$ satisfies.
(d) Let $V(n)$ be the number of vertices in $T_n$. Find a recurrence that $V(n)$ satisfies.
(e) Let $E(n)$ be the number of edges in $T_n$. Find a recurrence that $E(n)$ satisfies.

2. **Modeling/Representing Problems as Graphs.** I have $10, and I plan to spend some or all of my money on three types of candy, which I will buy one piece at a time.

- Chocolate bars cost $3.
- Almond Roca cost $2.
- Caramel chunks cost $5.

I want to know what sequences of candy purchases I can afford to make. (In a given sequence of candy purchases, I might buy more than one candy of a given type.)

(a) Describe how you would model this situation using a directed graph, where paths in your graph should represent possible sequences of candy purchases. What are the vertices, and when are two vertices connected with an edge?
(b) Is this graph directed or undirected, and why?
(c) How can this graph be used to determine which amounts of change I might have left over when I have had my fill of candy?
(d) Is your graph a DAG? Explain why or why not.

3. **Graphs, Trees, Tours, and Counting.** A complete bipartite graph is an undirected graph where the vertex set $V$ can be partitioned into two disjoint subsets $V_1$ and $V_2$, where $V_1 \cup V_2 = V$, so that

- there is an edge between every (unordered) pair of vertices $(v_i, v_j)$ where $v_i \in V_1$ and $v_j \in V_2$,
- there are no edges between any pair of vertices in $V_1$, and
- there are no edges between any pair of vertices in $V_2$.

Answer the following questions about complete bipartite graphs, and explain how you came to your conclusions. (You’ll notice that this is similar to a homework problem...)

(a) How many vertices are there in a complete bipartite graph with $|V_1| = m$ and $|V_2| = n$?
(b) How many edges are there in a complete bipartite graph with $|V_1| = m$ and $|V_2| = n$?
(c) For which values of $m$ and $n$ is a complete bipartite graph an unrooted tree?
(d) For which values of $m$ and $n$ does a complete bipartite graph have an Eulerian tour that starts and ends at different vertices?
(e) For which values of \( m \) and \( n \) does a complete bipartite graph have a Hamiltonian tour?

(f) If \( m = n \) how many Hamiltonian tours does a complete bipartite graph have?

4. **Recursively Defined Graphs.** Let the directed graph \( G_1 \) be a single vertex, which we will give the name \( S_1 \) and \( L_1 \). (Here, and in the following, “S” is a label that is given to a source vertex, and “L” is a label that is given to a sink vertex.) Let \( G_2 = G_1 \). Then for \( n \geq 3 \), let \( G_n \) be constructed from \( G_{n-2} \) and \( G_{n-1} \) as follows. \( G_n \) has a source vertex \( S_n \), with one directed edge pointing to the vertex \( S_{n-2} \) of graph \( G_{n-2} \), and one directed edge pointing to the vertex \( S_{n-1} \) of graph \( G_{n-1} \). \( G_n \) also has a sink vertex \( L_n \), with one edge incoming from \( L_{n-2} \) and one edge incoming from \( L_{n-1} \). The graph \( G_4 \) is shown below for clarity.

To build graph \( G_4 \), we start with the leftmost vertex \( S_4 \), add an edge to the source of graph \( G_3 \) (the upper edge), and an edge to the start of graph \( G_2 \) (the lower edge). We then connect the sinks of \( G_3 \) and \( G_2 \) to the sink \( L_4 \) of graph \( G_4 \). The sink \( L_4 \) is the rightmost vertex in the figure.

(a) Draw the graph \( G_5 \).

(b) How many paths are there from the start of your graph, \( S_5 \), to the end of your graph, \( L_5 \)?

(c) Let \( P(n) \) be a recurrence relation for the number of paths from \( S_n \) to \( L_n \) in graph \( G_n \). Find a recurrence that \( P(n) \) satisfies.

(d) Let \( L(n) \) be the length of the longest path from \( S_n \) to \( L_n \) in graph \( G_n \). Find a recurrence that \( L(n) \) satisfies.

(e) **EXTRA. Goes beyond midterm practice.** Let an overloaded vertex be defined as a vertex with more incoming edges than outgoing edges, or more outgoing edges than incoming edges. Similarly, let a balanced vertex be defined as a vertex with the same number of incoming edges as outgoing edges. [For example, in graph \( G_4 \), we have 4 overloaded vertices and 3 balanced vertices.] Let \( OV(n) \) be the number of overloaded vertices in \( G_n \). Find a recurrence that \( OV(n) \) satisfies. Similarly, let \( B(n) \) be the number of balanced vertices in \( B(n) \). Find a recurrence that \( B(n) \) satisfies.

5. **Counting: Poker Hands.** In each of the following problems, a hand of five cards will be dealt from a standard deck of cards, with thirteen cards in each of four suits, and no jokers or wild cards. A hand is a set of five cards, so the order in which the cards are dealt does not matter. Say how many different hands of the following types are possible. Here, as well as on the exam, you can leave your answer as an unsimplified algebraic expression involving binomial coefficients, factorials, or exponents. You do not need to simplify.

(a) Straight: A hand where the numbers of the cards are five consecutive integers (with Jack = 11, Queen = 12, King = 13, and Ace counting as 1 or 14).

(b) Three of a kind: A hand with three cards of one number, one card of a second number, and one card of a third number.

(c) Two pair: A hand with two cards of one number, two cards of a second number, and one card of a third number.
6. Counting: New Age Poker Hands. In the 30th century, the game of poker has been reinvented. Instead of a 4-suit, 52-card deck, the new deck of cards has 5 suits and 65 cards. Poker hands are still made with five cards.

(a) There is a new poker hand - the overflush. This hand is made up of one card from each suit. How many ways are there to make an overflush? (Disregard the fact that an overflush can also be, e.g., a five-of-a-kind. Just count overflushes.)

(b) How many ways are there to make five-of-a-kind?

(c) Poker hands are ranked by probability (higher-ranked hands have a lower probability of being made). Is a full house (three cards of one value, and two cards of a different value) still ranked higher than a flush?

(d) Is the royal flush still the highest-ranked hand?

7. Counting: Digits and Numbers. Using the digits 1, 2, 3 and 5, how many 4-digit numbers can be formed if:

(a) The first digit must be 1 and repetition of digits is allowed?

(b) The first digit must be 1 and repetition of digits is not allowed?

(c) The number must be divisible by two and repetition of digits is allowed?

(d) The number must be divisible by two and repetition of digits is not allowed?


(a) Describe how to construct a DAG so that paths in the DAG correspond to increasing subarrays of $A$.

(b) Why is your graph a DAG?

(c) EXTRA: This part goes beyond midterm practice. Recall from lecture slides that a layering of a DAG $G$ is defined recursively as follows: Label all sources of the current $G$ as being on the next level. Then, remove these sources, as well as their outgoing edges, from $G$. Repeat the above until $G$ contains no more vertices. Use a layering of the DAG that you described in part (a) to prove that in any array containing 26 distinct integers, there must be six elements in the array that are either increasing from left to right or else decreasing from left to right. [Hint: 26 = 25 + 1. 6 = 5 + 1.]

9. Modeling, DAGs and Topological Orderings. You have a system of $n$ variables representing real numbers, $X_1, \ldots, X_n$. You are given a list of inequalities of the form $X_i < X_j$ for various pairs of indices $i,j$. You want to know whether you can deduce with certainty from the given information that $X_1 < X_n$. For example, suppose that $n = 4$. A possible input is the list of inequalities $X_1 < X_2; X_1 < X_3; X_4 < X_3; \text{and} \ X_3 < X_2$. Does it follow that $X_1 < X_4$?

(a) Give a description of a directed graph that would help solve this problem. Be sure to define both the vertices and edges in terms of the variables and known inequalities.

(b) Draw the graph you described for the example above. Does $X_1 < X_4$ follow? Why or why not?

(c) Say which algorithm from lecture we could use on such a graph to determine whether $X_1 < X_n$ follows from the known inequalities.

10. Modeling, DAGs and Topological Orderings. A ballet recital consists of some number of different acts, each containing several dancers. Suppose you are given a list of all the acts in the recital, with the names of the dancers that will appear in each act. Some dancers may be in more
than one act, but each act requires a different costume. To allow the dancers time to change costumes, the recital should be set up so that no dancer is in two consecutive acts. Our goal is to find an ordering of the acts for the recital so that no dancer is in back-to-back acts.

(a) Describe how to model this situation using a graph. Carefully describe what the vertices of your graph represent, and when two vertices are connected with an edge.

(b) Now that you have modeled the situation with a graph, which problem from graph theory are you trying to solve on this graph?

(c) Is it always possible to achieve the goal? Explain.

11. **DAGs and Trees.** For each statement, either prove that it is true or give a counterexample to show that it is false.

(a) Every tree with \(n\) vertices has exactly \(n - 1\) edges. (If you wish, you may assume as known facts that a tree with \(n\) vertices is connected and has no cycles.)

(b) Every graph with exactly \(n - 1\) edges is a tree.

12. **Counting: More...**

(a) How many 5-card hands can be formed from an ordinary deck of 52 cards if exactly two suits are present in the hand?

(b) In any bit string, the longest consecutive run length is the maximum number of consecutive 1’s or consecutive 0’s in the string. For example, in the string 110100111, the longest consecutive run length is 3. How many bit strings of length 10 have a longest consecutive run length of 6?

(c) A software company assigns its summer interns to one of three divisions: design, implementation, and testing. In how many ways can a group of ten interns be assigned to these divisions if each division requires at least one intern?

13. **Hamiltonian Tours.** Recall that a complete graph is a simple undirected graph where there is an edge between every pair of distinct vertices. For \(n, m \geq 1\), define a kite graph, \(Kite(n, m)\), to be a graph that is made up of a complete graph on \(n\) vertices, connected at one vertex to a chain containing \(m\) additional vertices. Thus, as illustrated in the example below, the kite graph \(Kite(4, 3)\) would have a total of seven vertices and nine edges.

(a) How many edges are there in the graph \(Kite(n, m)\)?

(b) How many Hamilton tours does the graph \(Kite(4, 3)\) have?

(c) How many Hamilton tours does the graph \(Kite(n, m)\) have?
14. **DAGs and Topological Orderings.**

How many possible topological orderings exist for the following graph:

```
1 ——— 2 ——— 3
     |         |
     v         v
4 ——— 5
```

15. **Trees and DAGs.**

(a) Draw an unrooted tree with at least three vertices that has an Eulerian tour.
(b) Draw an unrooted tree with at least three vertices that has no Eulerian tour.
(c) Draw a rooted tree with exactly 3 internal vertices and exactly 10 leaves, or explain why no such tree exists. Recall: the root counts as an internal vertex.
(d) Draw a binary rooted tree with height exactly 2 and exactly 3 leaves, or explain why no such tree exists.
(e) Draw a binary rooted tree with height exactly 2 and exactly 5 leaves, or explain why no such tree exists.

16. **Trees and DAGs, Yet Another!** Consider a graph where the vertices are the numbers from 1...N for some odd number N, and we put an edge from each i ≤ N/2 to 2i and 2i + 1. For each question, give a yes or no answer and a short explanation.

(a) Is this graph a DAG?
(b) Is it a rooted tree?
(c) If yes, what is the root of the tree, and what is the parent of vertex i?
(d) If no, give a reason.

17. **Modeling/Representation With Graphs: Role-Playing Games.** In a role-playing game, our character at all times carries exactly one item. When our character offers the item to some non-player characters (NPC’s), and it is one they are interested in, they will trade it for your choice of items they own.

Suppose that you know, for each NPC, which items they are interested in and which items they own.

(a) Describe a method to construct a graph whose paths represent sequences of trades we could make.
(b) What graph algorithm could you use to determine whether there’s a series of trades that swaps one item for another?
(c) Draw your graph for the following example, and suppose you start out with some herbs that you found.
   i. The Brewer wants herbs and has mead
   ii. The Knight wants a potion, the queen’s favor, or a sword, and has a dragonscale
   iii. The Merchant wants gold, and has herbs and silk
   iv. The Queen wants silk or a dragonscale, and has the queen’s favor and gold
v. The Smith wants mead or gold, and has a sword
vi. The Witch wants herbs or a dragonscale, and has a potion

18. **Modeling/Representation With Graphs: Wireless Hotspots.** You are given a map of the wireless hotspots around your neighborhood. On this map, each hotspot’s coverage area is indicated by a shaded region surrounding that hotspot. You want to plan a walk to complete your errands in your neighborhood that stays completely within shaded regions so that you have continuous, unbroken wireless access.

(a) Describe how to model this situation using an undirected graph, where paths in your graph should correspond to walks that stay within shaded regions. What are the vertices, and when are two vertices connected with an edge?

(b) In terms of wireless internet access, what would it mean for this graph to be connected? Briefly justify your answer.
19. Addendum: Additional Counting Practice

(a) For security reasons, a credit card company wants to give out only credit card numbers that can be quickly checked for authenticity. It decides to use credit card numbers that satisfy the following properties:

i Each credit card number must be 16 digits long.
ii Each credit card number must end in an odd number or have identical first and second digits (or both).

For example, 8989 2567 4011 2234 is not a valid credit card number. How many valid credit card numbers are possible under this system? You can leave your answer as an unsimplified algebraic expression involving binomial coefficients, factorials, or exponents. You do not need to simplify, but show all work leading to your answer.

(b) For each of the parts in this question, you may leave your answer as an unsimplified algebraic expression involving binomial coefficients, factorials, or exponents. You do not need to justify your answers for this question.

i How many length 7 strings of lower case letters consist of two different letters alternating? For example: “yayayay”?
ii How many length 7 strings of lower case letters contain the letter “a” at least once? For example: “yuueicy”.
iii How many length 7 strings of lower case letters are composed of seven distinct letters arranged in lexicographic (dictionary) order?

(c) For the following problems, you may leave your answer in terms of binomial coefficients, factorials, and exponents. You do not need to simplify.

i How many rearrangements are there of the letters in APPEAR?
ii How many length 4 strings with distinct letters can be made from the letters in APPEAR?
iii How many length 3 strings with distinct letters can be made from the letters in APPEAR?

(d) For the following problems, you may leave your answer in terms of binomial coefficients, factorials, and exponents. You do not need to simplify.

i Suppose there are 7 roommates living together in a house. The dishes must be done once every day of the week. How many ways can the 7 roommates be assigned to do the dishes if each roommate does the dishes at least once per week?
ii Suppose there are 4 roommates living together in a house. The dishes must be done once every day of the week. How many ways can the 4 roommates be assigned to do the dishes if each roommate does the dishes at least once per week and no more than twice per week?

(e) Suppose you are a photographer at a family reunion with 10 family members (including Grandma and Grandpa).

i In how many ways can you line up 6 of the 10 family members to take a picture? No justification needed.
ii In how many ways can you line up 6 of the 10 family members to take a picture, such that Grandma is in the picture? No justification needed.
iii In how many ways can you line up 6 of the 10 family members to take a picture, such that Grandma and Grandpa are both in the picture? No justification needed.