Homework 8 Solutions

1) Which of the following graphs are isomorphic?

a) Degree Sequence: 2, 3, 3, 3, 3, 5, 5

b) Degree Sequence: 2, 2, 3, 4, 4, 4, 5

c) Degree Sequence: 1, 2, 3, 4, 4, 5, 5

d) Degree Sequence: 2, 2, 3, 4, 4, 4, 5

Graphs b and d have the same degree sequences. A and c do not share the same degree sequences with another graph so they cannot be isomorphic. b and d also have the same number of vertices and edges, and the same number of connected components. Thus, b and d are isomorphic.
(2) Which of the following degree sequences are possible for a simple graph?

a) 2, 1, 2, 5, 1

This degree sequence has 5 nodes, so a given node cannot have a degree more than 4. This degree sequence is therefore not possible because a node has a degree of 5.

b) 5, 5, 7, 1, 8, 1, 1, 4, 3

This degree sequence has 9 nodes, so a given node cannot have a degree more than 8. This seems to be true. However, let's try adding the degrees together:

$$5 + 5 + 7 + 1 + 8 + 1 + 1 + 4 + 3 = 35.$$ 

For a graph, the sum of the degrees is twice the number of edges. This implies our graph would have $$\frac{35}{2}$$ edges, but we cannot have half an edge. Thus, this degree sequence is not valid.

c) 3, 3, 4, 3, 3

This is a valid degree sequence.

Example
d) 9, 9, 5, 4, 5, 6, 8, 5, 7, 8

This is a valid degree sequence. Here is an example graph:
Which of the following have Eulerian circuits or Eulerian trails?

a) This graph is not connected. Thus, it cannot have an Eulerian trail or Eulerian circuit, because there is no way to get from the subgraph formed by the nodes L, M, and N, to the subgraph formed by the nodes F, G, H, I, J, and K.

b) The degree sequence for this graph is 2, 3, 4, 5, 5, 6, 6, 7, 8. There are exactly two nodes with odd degree, thus this simple connected graph has an Eulerian trail. It does not have an Eulerian circuit, however, because there is at least one vertex of odd degree.
The degree sequence for this graph is 2, 2, 4, 4, 4, 6, 6, 6.

Since all of the degrees are even, this simple connected graph has both an Eulerian trail and an Eulerian circuit.

The degree sequence for this graph is 2, 2, 3, 3, 3, 5.

There are more than two vertices with an odd degree, so this graph cannot possibly have an Eulerian trail or an Eulerian circuit.
Consider the graph given below. Give an Euler circuit through the graph by listing the vertices in the order visited.

The degree sequence for this graph is 2, 2, 2, 4, 4. Since all degrees are even, there must be an Euler circuit.

Since we are looking for a circuit, we can start with any node.

Let's start with K because it has a degree of 2. This slightly simplifies the problem because we know that we cannot revisit K until the end. Let's pick edge C first. Then it does not really matter which edge we pick next, so let's pick edge G. This makes our circuit look like this so far:

Next, we can't pick edge D yet, otherwise we will finish the path without covering all of the edges. Since we can pick from B or F, but the choice doesn't matter, let's just pick edge B. This forces the next edges to be A, then E, then F, then D. So, the listing of the vertices is:

K, M, N, L, M, J, N, K.

*Note: This is not the only possible listing!
which of the following have hamiltonian circuits?

a)

In a hamiltonian circuit, we need to visit every vertex exactly once and also make it back to our starting node.

Since this graph has a vertex with degree 1, (vertex E) there cannot possibly be a circuit. This is because when we try to visit vertex E, we cannot leave it again without going back through the vertex we came from. This would violate the condition that we visit each vertex exactly once. Thus, this graph does not have a hamiltonian circuit.

b)

This graph has a hamiltonian circuit.
This graph does not have a Hamiltonian circuit. Let's start at vertex T. There are only two directions we can go, to Q or to V. Let's assume we go to Q from T.

This means our second-to-last vertex visited will have to be V, otherwise we won't end up back at T. We can continue working backwards. From V, we cannot come from Q (since Q is the second vertex we visit) so we must come from X. Similarly, to get to X, we cannot come from V or Q, so we must have come from Y. To get to Y, we must have come from Q, which would make the cycle: T, Q, Y, X, V, T. But this does not cover all the vertices. So, a Hamiltonian circuit is not possible in this graph, because we cannot form one starting at vertex T.

d)

Yes there is a Hamiltonian circuit.
What is the chromatic number of each graph?

a)

The chromatic number is the smallest number of colors required to color the graph. The chromatic number for this graph is 3. Let's start with vertex A. Let's give it the color $C_1$. Then, the colors of B, E, and G must be a different color. Let's try coloring the graph with only two colors: $C_1$ and $C_2$:

If A is colored with $C_1$, then B must be colored with $C_2$, which means H has to be colored with $C_1$. Then, F has to be colored with $C_2$, and C must be colored with $C_1$.

This is not a valid coloring because vertices A and E are connected, but they are given the same color. Next, we can try using 3 colors, $C_1$, $C_2$, and $C_3$. Let's color A similarly to how we did the 2-coloring, but let's color vertex F with $C_3$. Then we can color vertex C with $C_1$, which means D must be colored with $C_3$. Then J must be $C_2$, G must be $C_3$. 

Therefore, the chromatic number of this graph is 3.
and I must be $C_2$.

b)

The Chromatic number for this graph must be at least 3 since the color for $A$ must be different than the color for vertex $B$ and vertex $C$, and the color for $B$ and $C$ must be different. So let's try to color the graph with 3 colors $C_1$, $C_2$, and $C_3$. This coloring worked so the Chromatic number is 3.
C) For this graph, start with color 1, \( C_1 \), and alternate colors as you go along the walk. On the last node, if it will end up being different than the first vertex, then the chromatic number is 2. In this case, the last vertex would be the same as the first, so we color the last vertex as \( C_3 \).

d) The first thing to notice is that this is a bipartite graph. This means we can color the "top row" one color, and the "bottom row" a second color. Thus, the chromatic number is 2.
Consider the graph below. Use the greedy algorithm to find the minimum spanning tree.

Let's use Prim's algorithm for this problem, and we will start with node J (you can start with any node). The smallest weighted edge between \{J\} and \{B, C, D, E, F, G, H, I, J\} is the edge JF. This is our first edge, and our spanning tree has the vertices \{F, J\}.

The next smallest edge between \{F, J\} and \{B, C, D, E, G, H, I, J\} is the edge GJ, with weight 2.

The next smallest edge between \{F, G, J\} and \{B, C, D, E, H, I, J\} is the edge FI, with weight 4.
Continue this process until you have reached all the vertices.

Minimum Spanning Tree
a) The weight of the minimum spanning tree is
\[1 + 2 + 3 + 4 + 6 + 7 + 9 + 10 = 42\]

b) If you used Prim's algorithm, the weights in order of selection is: 1, 2, 4, 6, 3, 7, 9, 10

If you used Kruskal's algorithm, the weights in order of selection is: 1, 2, 3, 4, 6, 7, 9, 10
Consider the graph given below. Use the greedy algorithm to find the minimum spanning tree.

For this problem, we will use Kruskal's algorithm. For this algorithm we go through the smallest weighted edges in increasing edge weight, and add edges whose corresponding vertices have not been added to the minimum spanning tree. We stop when all vertices are in the MST.