1. If $A$ and $B$ are events in a probability space with $Pr(A) = \frac{2}{5}$, $Pr(B) = \frac{1}{5}$ and $Pr((A \cap B)^c) = \frac{5}{6}$, then what is $Pr((A \cup B)^c)$?

Solution. Since $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$ by inclusion/exclusion and we are given that $Pr((A \cap B)^c) = \frac{5}{6}$, $Pr(A) = \frac{2}{5}$, $Pr(B) = \frac{1}{5}$, then $Pr(A \cap B) = 1 - \frac{5}{6} = \frac{1}{6}$ so

$$Pr(A \cup B) = \frac{2}{5} + \frac{1}{5} - \frac{1}{6} = \frac{13}{30}$$

and $Pr((A \cup B)^c) = 1 - Pr(A \cup B) = 1 - \frac{13}{30} = \frac{17}{30}$.

2. 7 hamsters and 8 rabbits are distributed randomly to 4 students. What is the probability that each student got at least one pet of each species?

Solution. By “Bars and Stars”, the total number of ways that the hamsters can be distributed is $\binom{7+3}{3}$ and the total number of ways that the rabbits can be distributed is $\binom{8+3}{3}$. Hence, the total number of ways that all the pets can be distributed is (by the product rule) equal to $\binom{10}{3}\binom{11}{3}$. However, if we require that each student get at least one pet of each specie, then we distribute these pets (both hamster and rabbit) first, leaving just $7 - 4 = 3$ “free” hamsters and $8 - 4 = 4$ “free” rabbits. Now, using Bars and Stars, we find that there are $\binom{3+3}{3}\binom{4+3}{3}$ ways to distribute these remaining jellybeans. Since we are assuming that all the distributions are equally likely, then the probability we want is

$$\frac{\binom{6}{3}\binom{7}{3}}{\binom{10}{3}\binom{11}{3}}.$$ 

3. How many 5-card hands can be formed from an ordinary deck of 52 cards if exactly two suits are present in the hand?
Solution. You have to be careful here not to overcount! There are \( \binom{4}{2} = 6 \) ways to choose the 2 suits. For example, suppose you choose ♠’s and ♥’s. Then possible hands can have 4♠’s and 1♥, or 3♠’s and 2♥’s, 2♠’s and 3♥’s, or 1♠ and 4♥’s. In the first case, there are \( \binom{13}{4} \binom{13}{1} \) possible hands that have 4♠’s and 1♥. Computing the number of possible hands for the other combinations in the same way, we see that the total number of possible hands (by the sum rule) is 6(\( \binom{13}{4} \binom{13}{1} \)+\( \binom{13}{3} \binom{13}{2} \)+\( \binom{13}{2} \binom{13}{3} \)+\( \binom{13}{1} \binom{13}{4} \)).

If the question asks for 7-card hands, the same reasoning is used, except then the possible hands can have 4♠’s and 3♥, or 3♠’s and 4♥’s.

4. A shelf contains 20 books. How many ways can 5 books be selected from these 20 with the restriction that no two selected books were adjacent before the selecting began?

Solution.

There are 20 − 5 = 15 unselected books. They form 15 + 1 = 16 places that the selected books can be at (i.e. 14 places between any two adjacent unselected books, plus one place at each end). Each way of choosing five of these 16 places corresponds to a way of selecting five books out of 20. Therefore, the answer is \( C(16, 5) \).

5. A hand \( H \) of 7 cards is chosen randomly from a standard deck of 52. Let \( E_1 \) be the event that \( H \) has at least one King and let \( E_2 \) be the event that \( H \) has at least 2 Kings. What is the conditional probability \( Pr(E_2 \mid E_1) \)?

Solution. The easy way to do this problem is to consider the following picture. Thus, the number of hands \( H \) with \( \geq 1 \) King is \( \binom{52}{7} - \binom{48}{7} \) and all these hands are equally likely. Among these, the number with exactly one King is \( 4 \binom{48}{6} \).

Therefore, \( Pr(H \text{ has exactly one King } \mid H \text{ has } \geq 1 \text{ King}) = \frac{4 \binom{48}{6}}{\binom{52}{7} - \binom{48}{7}} \).

What we want is the complement, namely
\[
Pr(H \text{ has } \geq 2 \text{ Kings } \mid H \text{ has } \geq 1 \text{ King}) = Pr(E_2 \mid E_1) = 1 - \frac{4 \binom{48}{6}}{\binom{52}{7} - \binom{48}{7}}.
\]

A more brute force way of doing this problem is to count directly the number of various kinds of hands. In particular, the number of hands \( H \) which have exactly \( k \) Kings is just
Figure 1: A partition of all $\binom{52}{7}$ hands.

$$
\binom{4}{k} \binom{48}{7-k} \text{ for } k = 1 \cdots 4. \text{ Thus, }
$$

$$
Pr(H \text{ has } \geq 1 \text{ King}) = \frac{1}{\binom{52}{7}} \sum_{k=1}^{4} \binom{4}{k} \binom{48}{7-k}.
$$

Similarly,

$$
Pr(H \text{ has } \geq 2 \text{ Kings}) = \frac{1}{\binom{52}{7}} \sum_{k=2}^{4} \binom{4}{k} \binom{48}{7-k}.
$$

Thus,

$$
Pr(E_2 | E_1) = \frac{Pr(H \text{ has } \geq 2 \text{ Kings and } H \text{ has } \geq 1 \text{ King})}{Pr(H \text{ has } \geq 1 \text{ King})} = \frac{Pr(H \text{ has } \geq 2 \text{ Kings})}{Pr(H \text{ has } \geq 1 \text{ King})} = \frac{\sum_{k=2}^{4} \binom{4}{k} \binom{48}{7-k}}{\sum_{k=1}^{4} \binom{4}{k} \binom{48}{7-k}}.
$$

These two different looking expressions for $Pr(E_2 | E_1)$ have the same value (of course!).

6. Prove by induction that $2^n > n^3$ for $n \geq 10$.

Solution. Prove by induction that $2^n > n^3$ for $n \geq 10$.

*Base case:* For $n = 10$, $2^n = 2^{10} = 1024$ and $n^3 = 1000$. Hence, the formula holds true.
**Induction Hypothesis:** Now, assume that this formula holds for some \( n \). That is, \( 2^n > n^3 \). Next, we need to prove that the formula holds true for \( n + 1 \).

**Inductive Step:** By hypothesis, \( 2^n > n^3 \). It is easy to see that \( n^3 > 3n^2 + 3n + 1 \) if \( n \geq 10 \). Now, \( 2^{n+1} = 2^n + 2^n > n^3 + 3n^2 + 3n + 1 = (n + 1)^3 \). Hence, the formula is true for \( n + 1 \).

7. A fair coin is flipped 3 times resulting in the flip sequence \( F_1 F_2 F_3 \). Consider the three events:
   (i) \( E_1 = \{F_1 \text{ is Heads}\} \);
   (ii) \( E_2 = \{F_2 \text{ and } F_3 \text{ agree}\} \);
   (iii) \( E_3 = \{F_1 \text{ and } F_3 \text{ disagree}\} \).

Which of the 3 pairs of events \( E_1 E_2 \), \( E_1 E_3 \) and \( E_2 E_3 \) are independent?

**Solution.** All three pairs are independent. To see this, it is enough to tabulate the \( E_i \):

\[
E_1 = \{HHH, HHT, HTT, HHT, HTT, TTT\},
E_2 = \{HHH, THH, HTT, TTT\},
E_3 = \{HHT, HTT, THH, TTH\}.
\]

Thus, \( E_1 \cap E_2 = \{HHH, HTT\} \), \( E_1 \cap E_3 = \{HHT, HTT\} \), \( E_2 \cap E_3 = \{HTT, THH\} \). Consequently, all the \( E_i \) have probability \( \frac{1}{2} \) and all the intersections have probability \( \frac{1}{4} \). So, since \( Pr(E_i \cap E_j) = \frac{1}{4} = Pr(E_i)Pr(E_j) \) for \( i \neq j \), then all the pairs are independent.

8. An jar contains \( m \) M&M’s and \( r \) Reese’s pieces. A random candy \( C_1 \) is drawn and a fair coin is flipped. If the flip is Heads then \( C_1 \) is put back into the jar. On the other hand, if the flip is Tails, the candy \( C_1 \) is not put back into the jar. Now another random candy \( C_2 \) is drawn from the jar.

   (i) What is \( Pr(C_2 = \text{M&M})? \)
   (ii) What is \( Pr(C_1 = \text{M&M}|C_2 = \text{M&M})? \)
   (iii) What is \( Pr(\text{Flip is Heads}|C_2 = \text{Reese’s})? \)

**Solution.**

There are two ways (at least!) to tackle this problem. The first way is a brute force approach which we will now demonstrate. Let’s first construct the decision tree for this process.

Now it is just a matter of reading off the appropriate probabilities from the decision tree. So, \( Pr(C_2 = \text{MM}) = \frac{m}{2m+r} + \frac{m}{2m+r} \frac{m-1}{m-1+r} + \frac{r}{2m+r} \frac{m}{m-1+r} \). However, this expression simplifies to give \( Pr(C_2 = \text{MM}) = \frac{m}{m+1}! \) Similarly, we get \( Pr(C_1 = \text{MM}|C_2 = \text{MM}) = \frac{r}{m+1} \frac{m}{m+r} \frac{m-1}{m-1+r} = \frac{1}{2} \frac{m-1}{m-1+r} \). Finally, since \( Pr(C_2 = \text{Reese}) = 1 - \frac{m}{m+r} = \frac{r}{m+r} \) then \( Pr(\text{Flip is Heads}|C_2 = \text{Reese}) = \frac{1}{2} \).

9. How many different way are there of arranging all the letters of
the string TEXASINSTRUMENTS?

**Solution.** The 16 letters from our word are $AE^2IMN^2RS^3T^9UX$. Thus, the number of rearrangements is $\frac{16!}{2!2!3!3!}$.

10. What is the coefficient of $x^4$ in the expansion of $(3x - 1)^9$?

**Solution.** By the binomial theorem, the coefficient of $x^k$ in the expansion of $(ax + b)^n$ is $\binom{n}{k}a^kb^{n-k}$. Thus, for our problem, the desired coefficient is $\binom{9}{4}3^4(-1)^5 = -\binom{9}{4}3^4$.

11. It is known that with probability $\alpha$ a randomly selected person doesn’t have tuberculosis, and with probability $1 - \alpha$, a randomly selected person does. There is a test $T$ which behaves as follows. If $T$ is applied to a healthy person then with probability $\beta$, it says that the person is healthy (so with probability $1 - \beta$, $T$ says that the person has tuberculosis). On the other hand, if $T$ is applied to a person with tuberculosis then with probability $\gamma$, it says they are sick (and with probability $1 - \gamma$, it says that they are healthy).

What is the probability that a randomly selected person is healthy given that the test $T$ says they have tuberculosis?

**Solution.** Hence, the probability we want is

$$Pr(\text{person is healthy } | \text{ test says sick}) = \frac{Pr(\text{person is healthy and test says sick})}{Pr(\text{test says sick})} = \frac{\alpha(1 - \beta)}{\alpha(1 - \beta) + (1 - \alpha)\gamma}.$$
12. Which of the following are true?
(i) \( f(n) = O(g(n)) \) implies \( f(n) = o(g(n)) \);
(ii) \( f(n) = o(g(n)) \) implies \( f(n) = O(g(n)) \);
(iii) \( f(n) = O(g(n)) \) and \( g(n) = O(h(n)) \) implies \( f(n) = O(h(n)) \);
(iv) \( 2^n = o(n!) \);
(v) \( x^2 \log x = O(\sqrt{x^4 + 100x^3}) \);

Solution.
(i) Take \( f(n) = n \) and \( g(n) = 2^n \), \( f(n) = O(g(n)) \) for \( x = 1 \) and \( M = 2 \). However, \( \lim_{n \to \infty} \frac{n}{2^n} = \frac{1}{2} \neq 0 \) which means that \( n \neq o(2n) \)
(ii) This is true trivially because \( f(n) = o(g(n)) \rightarrow f(n) \leq 1 \times g(n) \), so just pick \( M = x_0 = 1 \).
(iii) \( f(n) = O(g(n)) \) implies that there exists \( M_1 \) and \( x_1 \) s.t \( |f(n)| \leq M_1|g(n)| \) for all \( x > x_1 \).
\( g(n) = O(h(n)) \) implies that there exists \( M_2 \) and \( x_2 \) s.t \( |g(n)| \leq M_2|h(n)| \) for all \( x > x_2 \). We want to see if there exists an \( M_3 \) and \( x_3 \) s.t \( |f(n)| \leq M_3|h(n)| \) for all \( x > x_3 \). Indeed there is pick \( x_3 = \max(x_1, x_2) \) and \( M_3 = \max(M_1, M_2) \)
(iv) \( \lim_{n \to \infty} \frac{2^n}{n!} = \lim_{n \to \infty} \frac{2 \times 2 \times \cdots \times 2}{n(n-1) \cdots 1} = 0 \). Therefore, \( 2^n = o(n!) \)
(v) \( f^2 = O(g^2) \) is equivalent to \( f = O(g) \), therefore let’s take \( f^2(n) = x^4 \log^2 x \) and \( g = x^4 + 100x^3 \). \( f^2 \neq o(g^2(n)) \) and therefore \( f \neq O(g(n)) \)

(a) What is the expected number of draws until this happens?
(b) What is the answer if instead you draw with replacement?
Solution. Draw a decision tree. Notice the interesting fact that the probabilities at each of the terminal Red branches on the left are all equal to \( \frac{1}{n+1} \) (as we take the products of the probabilities going down to the end of the branch). Since the first branch takes just 1 step, and the second branch takes 2 steps, and in general, the \( k^{th} \) branch takes \( k \) steps, and there are altogether \( n+1 \) branches, then the expected value for the number of steps taken is just

\[
\frac{1}{n+1} \sum_{k=1}^{n+1} k = \frac{1}{n+1} \binom{n+2}{2} = \frac{n+2}{2}.
\]

Let \( E(m) \) denote the expected number of steps required if we have \( m \) White balls and 1 Red ball. Thus, \( E(0) = 1 \).

Thus, we have the recurrence \( E(n) = 1 \cdot \frac{1}{n+1} + (1 + E(n-1)) \cdot \frac{n}{n+1} \) (where the +1 in the factor \( 1 + E(n-1) \) comes from the fact that we took 1 additional step to get to that point). Multiplying both sides by \( n+1 \) and substituting \( F(n) = (n+1)E(n) \) for all \( n \), we have the simple (!) recurrence for \( F \), namely \( F(n) = n + 1 + F(n-1) \), \( F(0) = 1 \cdot E(0) = 1 \). The solution to this recurrence is \( F(n) = \binom{n+2}{2} \) so we find that \( E(n) = \frac{1}{n+1} \binom{n+2}{2} = \frac{n+2}{2} \).

For the case when we draw with replacement, the decision tree is actually infinite. However, with \( E^*(n) \) denoting the expected number of steps taken in this case, then the same recursive analysis as above gives the recurrence \( E^*(n) = 1 \cdot \frac{1}{n+1} + (1 + E^*(n)) \cdot \frac{n}{n+1} \). However, this directly implies that \( E^*(n) = n + 1 \). Thus, we can expect to wait about twice as long when drawing with replacement compared to drawing without replacement. An interesting problem is to see what happens in these two cases when we start with 2 (or \( k \)) Red candies and \( n \) White candies.

14. A sequence is defined by: \( a(1) = 1 \) and \( a(n+1) = 4a(n) - 1 \) for \( n \geq 1 \). What is \( a(100) \)?
Solution. The first few terms of \( a(n) \) are 1, 3, 11. The general solution to the homogeneous form of the recurrence is \( a(n) = c \cdot 4^n \). A specific (constant) solution \( \alpha \) must satisfy \( \alpha = 4 \alpha - 1 \), so we find \( \alpha = \frac{1}{3} \). Now, assuming the general solution to the complete recurrence has the form \( a(n) = c \cdot 4^n + \frac{1}{3} \), and substituting \( n = 1 \), we see that \( a(1) = 1 = c \cdot 4 + \frac{1}{3} \), so \( c = \frac{1}{6} \). Thus, the solution we want is \( a(n) = \frac{1}{6} \cdot 4^n + \frac{1}{3} \). (Check that this does generate the first few terms.) \( a(100) = \frac{1}{6} \cdot 4^{100} + \frac{1}{3} \).

15. How many sequences of length \( n \) made up of 1, 2 and 3 do not have two consecutive repeated symbols? (For example, 1231213231 would be allowed but 121123213212 would not.)

Solution. Observe that at any point in generating such a sequence, say we have so far \( s_1 s_2 \ldots s_t \), then there are exactly two choices for the next symbol \( s_{t+1} \). Since any of 1, 2 or 3 can be the starting symbol, then the total number of sequences of length \( n \) is \( 3 \cdot 2^{n-1} \).

16. What is the general solution to the recurrence: \( t(n + 2) = 2t(n + 1) + 2t(n), n \geq 0 \), with \( t(0) = 0, t(1) = 1 \)?

\[
t(n) = ((1 + \sqrt{3})^n - (1 - \sqrt{3})^n)/(2\sqrt{3})
\]

17. What is the general solution to the recurrence: \( t_n = 2t_{n-1} - 2t_{n-2} + 2n - 10, n \geq 0 \), with \( t_0 = 1, t_1 = -3 \)?

\[
t(n) = (\frac{7}{2} + 3i)(1 + i)^n + (\frac{7}{2} - 3i)(1 - i)^n + 2n - 6
\]

18. What is the general solution to the recurrence: \( x(n + 2) = 3x(n + 1) - 2x(n), n \geq 0 \), with \( x(0) = 0, x(1) = 1 \)?

\[
x(n) = 2^n - 1
\]

19. What is the general solution to the recurrence: \( x(n + 2) = 6x(n + 1) - 9x(n) + 6n + 10, n \geq 0 \), with \( x(0) = 0, x(1) = 1 \)?

\[
x(n) = \frac{1}{2}(3n + 3^n(5n - 8) + 8)
\]

20. A laundry basket contains 10 Orange and 12 Green shirts. A random set \( S \) of 6 shirts are removed from the basket without replacement. What is the probability
that \(S\) contains at least 2 Orange shirts, given that \(S\) contains an Orange shirt? What is the probability that that \(S\) contains at least 2 Orange shirts, given that \(S\) contains a Green shirt?

---

**Solution.** This problem is very similar to Problem 5. Let’s do it the easy way. Consider a venn diagram.

Since there are just \(\binom{22}{6} - \binom{10}{6}\) ways of choosing 6 shirts with at least one Orange shirt, then the probability of choosing 6 shirts with exactly one Orange given that the selection is known to have at least one Orange is \(\frac{10\binom{12}{5}}{\binom{22}{6}}\). Therefore, the complementary event, namely that the selection has at least two Orange shirts, given that it is known that it has at least one Orange is \(1 - \frac{10\binom{12}{5}}{\binom{22}{6}}\).

For the second part we want \(Pr(S \text{ has } \geq 2 \text{ Orange } | S \text{ has } \geq 1 \text{ Green})\).
The desired probability is seen to be \( \frac{\binom{22}{6} - \binom{10}{6} - \binom{12}{6} - 10 \binom{12}{5}}{\binom{22}{6} - \binom{10}{6}} \).

21. In how many ways can 12 identical baseballs be distributed to 2 boys and 3 girls if no boy gets more than 1 baseball and every girl gets at least 1 baseball?

Solution. This is Bars and Stars with a twist. First, let’s distribute the 3 baseballs to the girls at the beginning, leaving 9 “free” baseballs to distribute. Now, we consider 4 cases which account for which boys get a baseball or not. Namely, if neither boy gets a baseball, then there are \( \binom{9+2}{2} \) ways to distribute the 9 free baseballs (3 girls implies 2 Bars). On the other hand, if just one of the boys gets a baseball, then there are 8 baseballs left to distribute to the girls and this can be done in \( \binom{8+2}{2} \) ways (but don’t forget that there two ways for this to happen, namely Boy_1 gets a baseball and Boy_2 doesn’t, or the other way around). Finally, if both boys get baseballs, then there are only 7 baseballs left to distribute to the girls and this can be done in \( \binom{7+2}{2} \) ways. Thus, the total number of ways is \( \binom{11}{2} + 2 \binom{10}{2} + \binom{9}{2} \).

22. A valid driver’s license number \( D \) consists of 4 characters taken from the sets of 26 letters \( \{A, B, C, \ldots, Z\} \) and 8 numbers from \( \{0, 1, 2, \ldots, 9\} \). However, \( D \) must have at least one number and at least one number, and furthermore, \( D \) cannot have both of the symbols \( O \) and 0 in it. How many valid DL numbers are there?

Solution. This problem was slightly hairier than we planned! Consider a Venn diagram
of the various sets. It is divided up into 8 disjoint regions formed by 4 ovals.

For example, the union of regions $A, C, E, G$ and $F$, which we’ll denote by $ACEGF$, consists of all the passwords which are missing the letter $O$. Thus, the cardinality of $ACEGF$ is $35^4$. Similarly, $BDEGF$ consists of all the passwords which are missing the number 0, and so also has cardinality $|BDEGF| = 35^4$. Furthermore, the passwords in $EGF$ are missing both the letter $O$ and the number 0, and so $|EGF| = 34^4$. Thus, there are $2 \cdot 35^4 - 34^4$ passwords in $ABCDEFG$ by inclusion/exclusion. Also, region $CE$ has all the passwords which have no letter so that $|CE| = 10^4$. Similarly, $DF$ has all the passwords which have no number so that $|DF| = 26^4$. Hence the number of passwords satisfying our conditions is just $2 \cdot 35^4 - 34^4 - 10^4 - 26^4$. (It is a good thing we didn’t have requirements on the number of upper case and lower case letters as well!).

23. Suppose there are $n$ cats and $m$ dogs. How many ways are there to form $k$ (nonempty) pet committees using all $m + n$ animals?

How many ways are there so that each committee has at least one cat?

How many ways are there so that each committee has at least one cat and at least one dog?

Solution. The number of ways to form exactly $k$ committees with all the cats and dogs is $S(m + n, k)$.

The number of ways to form exactly $k$ committees with all the cats is $S(n, k)$. Then, we must distribute the dogs among these $k$ committees (a committee can have 0 dogs!). There are $k^m$ ways to do this. Thus, the number of ways to form $k$ committees so that each committee has at least one cat is $S(n, k)k^m$.

The number of ways to form $k$ committees with at least one dog is $S(m, k)$. So, the number of ways to form $k$ committees with at least one cat and at least one dog is $S(n, k)S(m, k)k!$.

24. Show by mathematical induction that your solutions for #17, #18, and #19 are correct.

17. Base case: $n=2$. $t_2 = 2t_1 - 2t_0 + (2)(2) - 10 = -14$ and
\[ t(2) = (3.5 + 3i)(1 + i)^2 + (3.5 - 3i)(1 - i)^2 + 4 - 6 = -14 \]

Then by induction,
\[ t_n = 2((3.5 + 3i)(1 + i)^{n-1} + (3.5 - 3i)(1 - i)^{n-1} + 2(n - 1) - 6) - 2((3.5 + 3i)(1 + i)^{n-2} + (3.5 - 3i)(1 - i)^{n-2} + 2(n - 2) - 6) + 2n - 10 \]
\[ = 2(3.5 + 3i)((1 + i)^{n-1} - (1 + i)^{n-2}) + 2(3.5 - 3i)((1 - i)^{n-1} - (1 - i)^{n-2}) + 4n - 4 - 12 - 4n + 8 + 12 + 2n - 10 \]
\[ = 2(3.5 + 3i)((1 + i)^{n-2}((1 + i) - 1)) + 2(3.5 - 3i)((1 - i)^{n-2}((1 - i) - 1)) + 2n - 6 \]
\[ = (3.5 + 3i)(1 + i)^n \frac{2i}{(1+i)^2} + (3.5 - 3i)(1 - i)^n \frac{-2i}{(1-i)^2} + 2n - 6 \]
\[ = (3.5 + 3i)(1 + i)^n + (3.5 - 3i)(1 - i)^n + 2n - 6 \text{ and we are done.} \]

18. Base case: \( n=2 \): \( x(2) = (3)(1) - (2)(0) = 3 \) by recursive formula and \( x(2) = 4 - 1 = 3 \) by general solution

Then by induction \( x(n + 2) = 3(2^{n+1} - 1) - 2(2^n - 1) \)
\[ = \frac{3}{2}2^{n+2} - 3 - \frac{3}{2}2^{n+2} + 2 \]
\[ = \frac{3}{2}2^{n+2} - \frac{3}{2}2^{n+2} + 1 \]
\[ = 2^{n+2}(\frac{3}{2} - \frac{1}{2}) - 1 \]
\[ = 2^{n+2} - 1 \text{ and we are done.} \]

19. Base cases: \( x(0) = 0 = \frac{1}{2}(3 \cdot 0 + 3^0(5 \cdot 0 - 8) + 8) = \frac{1}{2}(0 + 1(-8) + 8) = \frac{1}{2}(0) = 0 \)
\( x(1) = 1 = \frac{1}{2}(3 \cdot 1 + 3^1(5 \cdot 1 - 8) + 8) = \frac{1}{2}(3 + 3(5 - 8) + 8) = \frac{1}{2}(3 + 9 + 8) = \frac{1}{2}(2) = 1 \)

Inductive step (does it work for \( x(n + 2) = 6x(n + 1) - 9x(n) + 6n + 10? \))
\[ x(n + 2) = 6x(n + 1) - 9x(n) + 6n + 10 \]
\[ x(n) = 6x(n - 1) - 9x(n - 2) + 6(n - 2) + 10 \]
\[ = x(n) = 6(\frac{1}{2}(3 \cdot (n - 1) + 3^{n-1}(5 \cdot (n - 1) - 8) + 8)) - 9(\frac{1}{2}(3 \cdot (n - 2) + 3^{n-2}(5 \cdot (n - 2) - 8) + 8)) + 6(n - 2) + 10 \]
\[ = 2(\frac{1}{2}(9 \cdot (n - 1) + 3^n(5 \cdot (n - 1) - 8) + 24)) - (\frac{1}{2}(27 \cdot (n - 2) + 3^n(5 \cdot (n - 2) - 8) + 72)) + 6(n - 2) + 10 \]
\[ = (\frac{1}{2}(18 \cdot (n - 1) + 3^n(10 \cdot (n - 1) - 16) + 48)) - (\frac{1}{2}(27 \cdot (n - 2) + 3^n(5 \cdot (n - 2) - 8) + 72)) + 6(n - 2) + 10 \]
\[ = \frac{1}{2}(18 \cdot (n - 1) + 3^n(10 \cdot (n - 1) - 16) + 48 - 27 \cdot (n - 2) - 3^n(5 \cdot (n - 2) - 8) - 72 + 12(n - 2) + 20) \]
\[ = \frac{1}{2}(18n - 18 + 3^n(10 \cdot (n - 1) - 16) + 48 - 27n + 54 - 3^n(5 \cdot (n - 2) - 8) - 72 + 12n - 24 + 20) \]
\[ = \frac{1}{2}(18n - 27n + 12n + 48 + 54 - 24 + 20 - 18 - 72 + 3^n(10 \cdot (n - 1) - 16) - 3^n(5 \cdot (n - 2) - 8)) \]
\[ \frac{1}{2}(3n + 8 + 3^n(10n - 26) - 3^n(5n - 18)) \]
\[ = \frac{1}{2}(3n + 8 + 3^n(10n - 26 - 5n + 18)) \]
\[ = \frac{1}{2}(3n + 8 + 3^n(5n - 8)) \]
Which is what we wanted to show.

25. Prove that the running time of Merge Sort is \( O(n \log n) \) using a decision tree.

Solution. The insight here comes by looking at the implicit recursion formula for merge sort. To sort a list of size \( n \) we need to sort two lists of size \( n/2 \) and then merge them which takes time \( n \). This gives us \( T(n) = T(n/2) + T(n/2) + n = 2T(n/2) + n \) (assuming that \( n \) is even). The base case of merge sort is when each subproblem we have to solve is of size 1. With a problem of size \( n \) we need \( \log n \) steps to turn it into a subproblem of size 1. (After \( k \) steps we have a subproblem of size \( n/2^k \). We stop dividing up when this size reaches 1. This gives us \( n/2^k = 1 \) or \( k = \log n \). Hence we stop dividing after \( \log n \) steps). We are doing \( n \) work to merge subarrays at every layer of the tree so the total running time is then simply \( O(n \log n) \)

26. How many choices do you have to put \( n \) indistinguishable objects into \( k \) distinguishable bins where every bin has to have at least \( p \) objects?

Solution. When you see \( n \) indistinguishable objects into \( k \) distinguishable bins you should immediately think Stars and Bars so \( \binom{n+k-1}{k-1} \) but now we’ve added one more constraint: each of the \( k \) bins should have at least \( p \) objects. To solve this problem we thus distribute \( p \) objects to each of the \( k \) bins for a total of \( pk \) bins and then remove them from consideration when applying the stars and bars formula. The answer is then simply \( \binom{n+k-pk-1}{k-1} \)

27. Propose an algorithm for 2-coloring (no two adjacent vertices can have the same color) a graph \( G \). What is the running time of your algorithm?

Solution. We use a greedy algorithm for this. Start with any vertex and color it \( C_1 \) and color all it’s neighbors \( C_2 \) and mark this vertex “explored” so that we don’t have to visit it again. Next, recursively we look at recently colored but ”unexplored” vertices one by one and color its neighbors with an opposite color and mark it explored. Stop when all the vertices are ”explored”.
The running time of this algorithm is $O(n + 2m) = O(n + m)$ for a graph of $n$ vertices and $m$ edges because we visit each vertex for coloring at most once and we look at each edge at most twice, once each while exploring the corresponding vertices.

28. What is the length of the Minimum Spanning Tree for the following weighted graph?

![Figure 7: A weighted graph.](image)

Solution. The length of any minimum spanning tree for this graph (and there is more than one) is 60.

![Figure 8: A minimum spanning tree for G](image)
29. What is the Big-Θ of the general solution to the following recurrence relations?

(a) \( T(n) = 3T\left(\frac{n}{2}\right) + n^2 \)
By the master theorem, \( \log_2 3 < 2 \) so:
\[
T(n) = \Theta(n^2)
\]

(b) \( T(n) = 5T\left(\frac{n}{2}\right) + n \)
By the master theorem, \( \log_2 5 > 1 \) so:
\[
T(n) = \Theta(n^{\log_2 5})
\]

(c) \( T(n) = 27T\left(\frac{n}{3}\right) + n^3 \)
By the master theorem, \( \log_3 27 = 3 \) so:
\[
T(n) = \Theta(n^3 \log n)
\]

(d) \( T(n) = 2T(n-1) + n^{100} \)
Substitute in to get:
\[
T(n) = 2[2[2T(n-3) + (n-2)^{100}] + (n-1)^{100}] + n^{100}
\]
Now generalizing:
\[
T(n) = \Theta(2^i T(n-i))
\]
Let \( T(0) = 1 \) then:
\[
T(n) = \Theta(2^n)
\]

(e) \( T(n) = T(n-1) + n^{12} \)
As you expand, this become \( \sum_{i=0}^{n} i^{12} \). It is known that this sum is \( n^{13} \) (see http://www.math.rutgers.edu/~erowland/sumsofpowers.html). Thus:
\[
T(n) = \Theta(n^{13})
\]

(f) \( T(n) = T(n-1) + 2 \)
Expand:
\[
T(n) = ((T(n-3) + 2) + 2) + 2
\]
Generalize:
\[
T(n) = T(n-i) + 2i
\]
If \( T(0) = 1 \) we get:
\[
T(n) = \Theta(2n)
\]
(g) \( T(n) = T(\log(n)) + 1 \)

There will be \( \log^*(n) \) levels (the number of times it takes to get \(< 1\) by repeated logs), each level adds 1 to the recurrence. Thus:

\[
T(n) = \Theta(\log^*(n))
\]

With:

\[
\log^* n := \begin{cases} 
0 & \text{if } n \leq 1; \\
1 + \log^*(\log n) & \text{if } n > 1 
\end{cases}
\]

See [http://en.wikipedia.org/wiki/Iterated_logarithm](http://en.wikipedia.org/wiki/Iterated_logarithm) for info on the \( \log^* \) function, however, don’t worry about knowing this function for the final.