Lecture 9 Notes

• Goals for Today
  – Probability
    • Random variables, expectation, variance
  – Distribution functions
  – Types of distributions

• Midterm this Thursday – closed book, no calculators
  – You are allowed 1 handwritten 8.5”x11” sheet of paper

• Review session tonight 6:30-8:30pm, Ledden Hall
  – ~10 practice questions (questions and solutions will be posted for those who can’t attend)
Likely Midterm Coverage

- ~9 questions, 75 minutes
- Total of 110/100 points possible = “traditional”
- Question types
  - #anagrams
  - #passwords
  - Order form indistinguishable objects, marked boxes
  - Poker probability
  - Hypergeometric distribution \( P(E(B,D,b,d)) \) CL Example 29
  - Geometric probability
  - Expectation
  - Counting \{with, without\} x \{order, repetition\}
  - Sequences and probabilities games in series, moves on a graph
  - Counting functions injections, surjections, bijections
Random Variables

- Definition. Given a probability space \((U, P)\),

- *Hang on …*

- **What is \(U\)?**
  - \(U\) = a finite sample space (i.e., a finite set)

- **What is \(P\)?**
  - \(P\) is a *probability function on \(U\) \( P : U \rightarrow \mathbb{R} \)
  - \(P(t) \geq 0 \) for all \(t \in U\)
  - \(\sum_{t \in U} P(t) = 1\)
  - For any event \(E \subseteq U\), \(P(E) = \sum_{t \in E} P(t)\) is the probability of \(E\)
Random Variables

• Definition. Given a probability space \((U, P)\), we say that the function \(X: U \rightarrow \mathbb{R}\) which associates elements of \(U\) with numerical values is a *random variable*

\[X \text{ associates numerical values with elements of the sample space}\]

• Example. Flip a fair coin four times. Some associated random variables:
  – \(X = \) number of heads that appear
  – \(Y = \) result of second flip
  – \(Z = \) number of switches from H to T or vice versa

• What are possible values of \(X\)?
• What are possible values of \(Y\)?
• What are possible values of \(Z\)? = Image(Z)
Distribution Function

• Distribution function of a r.v. is $f_X : \text{Image}(X) \to \mathbb{R}$

  $$f_X(t) = P(X^{-1}(t)) = P(X = t)$$

• This is a probability function on the set $\text{Image}(X)$

• $f_X$ is a non-negative function with range $\mathbb{R}$

• $\text{Coimage}(X) = \{X^{-1}(t) \mid t \in \text{Image}(X)\}$ is a partition of $U$

• $1 = P(U)$
  
  $$= \sum_{t \in \text{Image}(X)} P(X^{-1}(t))$$

  $$= \sum_{t \in \text{Image}(X)} P(X = t)$$

  $$= \sum_{t \in \text{Image}(X)} f_X(t)$$
Coin Example

• Flip a fair coin four times. Let r.v. $X = \#\text{heads}$

• $f_X(2) = P(\text{exactly two heads appear}) = ?$
  – $C(4,2) \cdot (1/2)^4 = 3/8$
  – Corresponds to subset of $U = \{H,T\}^4 : \{TTHH, THTH, \ldots \}$

• $f_X(0) =$

• $f_X(1) =$

• $f_X(3) =$

• $f_X(4) =$
Again…

• The probability distribution function of the r.v. $X$ is $f_X : \text{Image}(X) \rightarrow \mathbb{R}$

\[ f_X(t) = P(\{u \in U \mid X(u) = t\}) = P(X = t) \]
FN Example 15: Roll a Fair Die

• $U = \{1, 2, 3, 4, 5, 6\}$ and $P(i) = 1/6$ for all $i$.

• $X(t) = t$
  
  $$f_X(t) = P(X^{-1}(t)) = P(t) = 1/6 \text{ for } t = 1, 2, 3, 4, 5, 6$$

• $X(t) = 0$ when $t$ is even and $X(t) = 1$ when $t$ is odd
  
  $$f_X(0) = P(X^{-1}(0)) = P(\{2, 4, 6\}) = 1/2 \text{ and } f_X(1) = 1/2$$

• $X(t) = -1$ when $t \leq 2$ and $X(t) = 1$ when $t > 2$
  
  $$f_X(-1) = P(\{1,2\}) = 1/3 \text{ and } f_X(1) = 2/3$$
Cumulative Distribution Function

- The **cumulative distribution function** (CDF) is a function from the possible values of a random variable to the real numbers.

- The CDF of a real-valued random variable $X$ is given by the function $F_X(t) = P(X \leq t)$

- Example: Suppose $X$ is a random variable which takes values between 0 and 1 with equal probability. ($X$ is *uniformly distributed* on $[0,1]$.)
  - CDF: $F_X(t) = t$ for $0 \leq t \leq 1$
Binomial Distribution

- Definition: Fix $n$. Toss the same (potentially biased with bias $p$ for H) coin $n$ times.
- $U = \{(f_1, \ldots, f_n) \mid f_i \in \{H, T\}\}$
- $X(f_1, \ldots, f_n) = \text{number of H’s in the sequence}$

- $f_X(0) = (1-p)^n$
- $f_X(1) = C(n, 1)p(1-p)^{n-1}$
- $f_X(2) = C(n, 2)p^2(1-p)^{n-2}$
- $\ldots$
- $f_X(n-1) = C(n, n-1)p^{n-1}(1 - p)$
- $f_X(n) = p^n$

- The distribution of $X$ is called the binomial distribution
Hypergeometric Distribution

• Definition: Pick \( n \) objects out of \( N \) total objects, where there are a total of \( K \) good objects. Let the random variable \( X \) correspond to the number of good objects picked.

• \( U = \{\{x_1, ..., x_n\}\} \)

• \( X(\{x_1, ..., x_n\}) = \) number of good objects

• \( f_X(k) = P( \text{ k good out of the } n \text{ picked } ) \)
  \[ = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} \]

• The distribution of \( X \) is called the hypergeometric distribution
Mean or Expected Value

- Mean value = expected value of one sample from a random variable
  - i.e., “average”
- If the random variable $X$ can take values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$, then the mean or expected value of $X$, written $E[X]$, is

$$E[X] = \sum_{k=1}^{n} (p_k x_k)$$
Expected Value

- Suppose that the random variable $X$ takes on values 
  \{x_1=1, x_2=2, x_3=3, x_4=4, x_5=5\} with probabilities

  \begin{align*}
  p_1 &= P(X = x_1) = .05 \\
  p_2 &= P(X = x_2) = .05 \\
  p_3 &= P(X = x_3) = .2 \\
  p_4 &= P(X = x_4) = .3 \\
  p_5 &= P(X = x_5) = .4 \\
  \end{align*}

Then $E[X] = \sum_{k=1}^{5} (p_k x_k) =$

\begin{align*}
= (1)(.05) + (2)(.05) + (3)(.2) + (4)(.3) + (5)(.4) \\
= 3.95
\end{align*}
Variance  Def. 8, FN-24

• Definition: The variance of a random variable is

\[ \sigma_x^2 = \text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \]

== Expected value of \( X^2 \),

minus   (expected value of \( X \), squared

• Variance is how much \( X \) is likely to vary through its distribution.
Variance Example

- Again: $X$ takes values $\{x_1=1, x_2=2, x_3=3, x_4=4, x_5=5\}$ with probabilities:

  $p_1 = P(X = x_1) = .05$

  $p_2 = P(X = x_2) = .05$

  $p_3 = P(X = x_3) = .2$

  $p_4 = P(X = x_4) = .3$

  $p_5 = P(X = x_5) = .4$

\[
E[X^2] = \sum_{k=1}^{5} (p_k x_k^2) = (1)(.05) + (4)(.05) + (9)(.2) + (16)(.3) + (25)(.4) = 16.85
\]

\[
E[X] = 3.95 \text{ from before}
\]

\[
Var(X) = E(X^2) - (E(X))^2 = 16.85 - (3.95)^2 = 1.2475
\]
Example: Mean of Binomial Distribution

- Flip a fair coin four times. Let $X =$ number of heads

- $E[X] =$
  
  
  \begin{align*}
  0*P(X = 0) \\
  + 1*P(X = 1) \\
  + 2*P(X = 2) \\
  + 3*P(X = 3) \\
  + 4*P(X = 4)
  \end{align*}
  
  
  \begin{align*}
  &= 0*C(4,0) + 1*C(4,1)(.5)^4 + C(4,2)(.5)^4 + C(4,3)(.5)^4 + C(4,4)(.5)^4 \\
  &= (4+12+12+4) / 16 = 2
  \end{align*}
Example: Mean of Hypergeometric Distribution

- Consider picking 4 items from 10 total items, with 6 good items.
- \( N = 10; \ K = 6; \ n = 4 \)
- Let r.v. \( X \) = number of good items picked

\[
E[X] = 0 \times P(X=0) + 1 \times P(X=1) + 2 \times P(X=2) + 3 \times P(X=3) + 4 \times P(X=4)
\]

\[
= 0 + 1 \times \frac{C(6,1)C(4,3)}{C(10,4)} + 2 \times \frac{C(6,2)C(4,2)}{C(10,4)} + 3 \times \frac{C(6,3)C(4,1)}{C(10,4)} + 4 \times \frac{C(6,4)C(4,0)}{C(10,4)}
\]

\[
= \frac{504}{210} = \frac{12}{5} = 2.4
\]
FN Theorem 3: Linearity of Expectation

• Theorem: If X and Y are two random variables on the same sample space and if a and b are real numbers, then:

\[ E(aX+bY) = aE(X) + bE(Y) \]

• Example: Roll two dice. What is the expected value of their sum?
Expected Sum of Two Rolled Dice

• Define r.v.’s:
  – $Z =$ sum of two dice
  – $X =$ value of first
  – $Y =$ value of second

• What is $E[Z]$ ?

• What is $E(X) = E(Y)$ ?
Problems 9

• P9.1 There are 20 countries on a planet. Among any three of these countries, there are always two with no diplomatic relations. Prove that there are at most 200 embassies on this planet.

• P9.2 Every participant in a tournament plays with every other participant exactly once. No game is a draw. After the tournament, every player makes a list with the names of all players, who (a) were beaten by him, and (b) were beaten by the players beaten by him. Prove that the list of some player contains the names of all other players.
Distribution Function, in BW Notes

• Idea: combine random variable and probability function concepts → distribution function

• Definition 6 **FN-22**: Let $X : U \rightarrow \mathbb{R}$ be a random variable on a sample space $U$ with probability function $P$. For each real number $t \in \text{Image}(X)$, let $X^{-1}(t)$ be the inverse image of $t$. Define a function $f_X : \text{Image}(X) \rightarrow \mathbb{R}$ by $f_X(t) = P(X^{-1}(t))$. The function $f_X$ is called the probability distribution function (or, probability density function) of the random variable $X$. 
Distribution Function, in BW Notes

• Let $X : U \rightarrow \mathbb{R}$ be a random variable on a sample space $U$ with probability function $P$.

• For each real number $t \in \text{Image}(X)$, let $X^{-1}(t)$ be the inverse image of $t$.
  – i.e., the set of elements of $U$ whose image is $t$

• Define a function $f_X : \text{Image}(X) \rightarrow \mathbb{R}$ by $f_X(t) = P(X^{-1}(t))$.
  – $P(X^{-1}(t))$ is probability of set of events $E = \{e \mid X(e) = t\}$
  – This is just $P(X = t)$, i.e., the probability that r.v. $X$ takes the value $t$

• The function $f_X$ is called the probability distribution function (or, probability density function) of the random variable $X$. 
Fun (from Lecture 6)

• If two sets can be put into 1-1 correspondence, their cardinalities are equal.

• “Cardinality” of infinite sets: very interesting!

* See the posted handout by Prof. Gabriel Robins, UVA

“Elvis’s Hotel”: although it’s always crowded, you still can find some room ...

Elvis’s Hotel has Rooms 1, 2, 3, … Every room is occupied by a guest.

Room 1: occupied - Room 10^{38}: occupied - Etc.

– One person arrives. Can he be accommodated?

\[ |\mathbb{N} \cup \{Bob\}| >, =, < |\mathbb{N}| ? \]

– Five people arrive. Can they be accommodated?

– |\mathbb{N}| people arrive. Can they be accommodated?

\[ |\mathbb{Z}| >, =, < |\mathbb{N}| ? \]
Fun (from Lecture 6)
• If two sets can be put into 1-1 correspondence, their cardinalities are equal.
• Which set has larger cardinality:
  – R (reals) or (0,1) ? **SAME**
  – N (counting numbers) or N × N = Q⁺ (positive rationals) ?
    – N (counting numbers) or R⁺ (positive reals) ?
Reading – Keywords!

• Random variables
• Distribution function of r.v.
• Expectation
• Variance
• Linearity of expectation
• Binomial distribution, Hypergeometric distribution
• Poisson distribution
• Normal distribution
• Joint distribution
• Marginal distribution
• Independence