UCSD CSE 21, Spring 2014 [Section B00]

Mathematics for Algorithm and System Analysis

Lecture 8

Class URL: http://vlsicad.ucsd.edu/courses/cse21-s14/
Lecture 8 Notes

• Goals for this week: Unit FN
  – Functions
    • Permutations: cycle notation, composition
    • Counting functions
  – Probability
    • Random variables, expectation, variance
    • Probability distributions
    • Birthday paradox
    • Random graphs
  – Sterling numbers of the second kind
• “Basic Concepts and Notation” posted
• Generated questions on WeBWorK
  – Functions set is up
  – “Worked-out solutions” can be viewed in WeBWorK!
• Midterm next Thursday – closed book, no calculators
  – You are allowed 1 handwritten 8.5”x11” sheet of paper
  – **Review session Tuesday 4/29 6:30-8:30pm, Ledden Hall**
  – Practice MT questions are posted
    • solution set to follow
### Ordered? With Repetition?

*(from RRR’s office hours)*

<table>
<thead>
<tr>
<th></th>
<th>No Repetition (without replacement)</th>
<th>With Repetition</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Order Matters</strong></td>
<td></td>
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</tr>
<tr>
<td></td>
<td>$P(n,k)$ #k-lists from n-set</td>
<td>$n^k$ #k-lists from n-set</td>
</tr>
<tr>
<td></td>
<td>$P(</td>
<td>B</td>
</tr>
<tr>
<td><strong>Order Doesn’t Matter (select, choose)</strong></td>
<td>$C(n,k)$ #k-sets from n-set; binomial coefficients</td>
<td>$C(n + k – 1, k – 1)$ n indistinguishable objects in k marked boxes (≡ selecting one of k boxes, n times with repetition); order form (n hotdogs, k flavors)</td>
</tr>
</tbody>
</table>
Counting Functions, Injections

- How many functions are there from a 3-set to a 7-set? \(7^3\)
- How many functions are there from a 7-set to a 3-set? \(3^7\)
- How many injections are there from a 3-set to a 7-set? \(P(7,3) = \frac{7!}{4!} = 7 \cdot 6 \cdot 5\)
- How many injections are there from a 7-set to a 3-set? 0
Number of Ways to Partition a Set

Example 24 (Set partitions) A partition of a set $B$ is a collection of nonempty subsets of $B$ such that each element of $B$ appears in exactly one subset. Each subset is called a block of the partition. The 15 partitions of \{1, 2, 3, 4\} by number of blocks are

1 block: \{\{1, 2, 3, 4\}\}
2 blocks: \{\{1, 2, 3\}, \{4\}\} \{\{1, 2, 4\}, \{3\}\} \{\{1, 2\}, \{3, 4\}\} \{\{1, 3, 4\}, \{2\}\} \{\{1, 3\}, \{2, 4\}\} \{\{1, 4\}, \{2, 3\}\} \{\{1\}, \{2, 3, 4\}\} 
3 blocks: \{\{1, 2\}, \{3\}, \{4\}\} \{\{1, 3\}, \{2\}, \{4\}\} \{\{1, 4\}, \{2\}, \{3\}\} \{\{1\}, \{2, 3\}, \{4\}\} \{\{1\}, \{2, 4\}, \{3\}\} \{\{1\}, \{2\}, \{3, 4\}\} 
4 blocks: \{\{1\}, \{2\}, \{3\}, \{4\}\}

“Stirling numbers of the second kind”
- $S(n,k) =$ number of ways to partition \{1, \ldots, n\} into exactly $k$ nonempty subsets
- $S(4,1) = 1$
- $S(4,2) = 7$
- $S(4,3) = 6$
- $S(4,4) = 1$
Number of Ways to Partition a Set

• \( S(n,k) \) = number of ways to partition \{1, \ldots, n\} into exactly \( k \) nonempty subsets

• Observations:
  – \( S(n,1) = 1 \) \( \forall n \)
  – \( S(n,n) = 1 \) \( \forall n \)
  – \( S(n,2) = ? \)
    • \( 2^{n-1} - 1 \)
  – \( S(n,n-1) = ? \)
    • \( C(n,2) \)
### Number of Ways to Partition a Set

<table>
<thead>
<tr>
<th>Number of Blocks</th>
<th>Partitions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 block:</td>
<td>{{1, 2, 3, 4}}</td>
</tr>
<tr>
<td>2 blocks:</td>
<td>{{1, 2, 3}, {4}}  {{1, 2, 4}, {3}}  {{1, 2}, {3, 4}}  {{1, 3, 4}, {2}}  {{1, 3}, {2, 4}}  {{1, 4}, {2, 3}}  {{1}, {2, 3, 4}}</td>
</tr>
<tr>
<td>3 blocks:</td>
<td>{{1, 2}, {3}, {4}}  {{1, 3}, {2}, {4}}  {{1, 4}, {2}, {3}}  {{1}, {2, 3}, {4}}  {{1}, {2}, {3}, {4}}</td>
</tr>
<tr>
<td>4 blocks:</td>
<td>{{1}, {2}, {3}, {4}}</td>
</tr>
</tbody>
</table>

- $S(n,k)$ = number of ways to partition $\{1, \ldots, n\}$ into exactly $k$ nonempty subsets
- What is $S(5,2)$?
  - Add “5” to all $S(4,1)$ partitions into 1 blocks
  - Add “5” to all $S(4,2)$ partitions into 2 blocks
Number of Ways to Partition a Set

- $S(n,k) =$ number of ways to partition $\{1, \ldots, n\}$ into exactly $k$ nonempty subsets

- What is $S(5,3)$?
  - Add “5” to all $S(4,2)$ partitions into 2 blocks
  - Add “5” to all $S(4,3)$ partitions into 3 blocks
Number of Ways to Partition a Set

<table>
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<tr>
<td>1 block</td>
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<td>3 blocks</td>
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- $S(n,k) =$ number of ways to partition \{1, …, n\} into exactly $k$ nonempty subsets

- Recursion:

\[
S(n,k) = S(n-1, k-1) + k \cdot S(n-1, k)
\]
From Last Time

• How many functions from \{1, 2, 3, 4\} to \{a, b, c, d, e\} have exactly two elements in the image?

• Case 1: \(4 = 1 + 3\)

\[
C(4,1) \cdot C(5,1) \cdot C(3,3) \cdot C(4,1) = C(4,1) \cdot C(3,3) \cdot C(5,2) \cdot 2 = 80
\]
From Last Time

- Recall: How many functions from \{1, 2, 3, 4\} to \{a, b, c, d, e\} have exactly two elements in the image?

- Case 1: \(4 = 2 + 2\)

\[
\binom{4}{2} \cdot \binom{5}{2} = 60
\]

\[
= \binom{4}{2} \cdot \binom{5}{1} \cdot \binom{2}{2} \cdot \binom{4}{1} / 2 = 60
\]

picking (1,2) \(\rightarrow\) a (implying (3,4) \(\rightarrow\) d) is same as picking (3,4) \(\rightarrow\) d (implying (1,2) \(\rightarrow\) a)
Counting Surjections

- Previous example: $S(4,2) = 7$ partitions of $A$ into 2 blocks
- Multiply by $5 \cdot 4$ (= choosing 2 elements of image in $B$)
- $7 \cdot 5 \cdot 4 = 140 = 80 + 60$

- So … number of surjections from $|A|$ to $|B|$, given that $|A| \geq |B|$?

$$S(|A|,|B|) \cdot (|B|)!$$
A Probability Question

- What is the probability that a random function from \{1, 2, 3, 4\} to \{a, b, c, d, e\} has exactly two elements in its range?

- \(|5|^{\mid 4\mid} = 625\) total functions
- \(140 / 625 = 0.224\)
Sampling With, Without Replacement

• Given an urn with 5 marbles: 2 white and 3 red

• Draw one marble at random, record its color, then replace. Then draw a second one. What is the probability that the second one is red?

• Draw one out, record color, then throw away. Then draw a second one. If the first one is white, what is the probability that the second one is red?

• Draw one out, don’t look at color, throw it away and draw a second one. What is the probability that the second one is red?
Sampling With, Without Replacement

• Betting on a six-horse horse race.

• Buy six tickets, each on a different horse
  *Without replacement*

• Buy six tickets, each time randomly choosing a horse
  *With replacement*

• What is the probability that at least one of the tickets picks the winning horse?
Random Variables

• Definition. Given a probability space \((U, P)\), we say that the function \(X: U \rightarrow \mathbb{R}\) is a random variable
  – \(X\) associates numerical values with elements of the sample space

• Example. Flip a fair coin four times. Some associated random variables:
  – \(X =\) number of heads that appear
  – \(Y =\) result of second flip
  – \(Z =\) number of switches from \(H\) to \(T\) or vice versa

• What are possible values of \(X\)?
• What are possible values of \(Y\)?
• What are possible values of \(Z\)?

Each value of a given random variable corresponds to an event (subset of \(U\)).
Another Probability Question

- In a room with \( n \) people, what is the probability that some two people have the same birthday?
- Assume that all possible birthdays are equally likely, and there are only 365 of them.
- Let \( A \) be the event that some two people in the same room have the same birthday.
- \( P(A) = 1 - P(A^c) \)

\( A = \) some two people have the same birthday

\( A^c = \) no two people have the same birthday
Birthdays

- \( P(A) = 1 - P(A^c) \)
- Let \( B_i \) be the probability that person \( i \) has a different birthday than persons 1, 2, ..., \( i-1 \)
- Note: \( B_1 = 1 \)
Birthdays

Let’s try to calculate \( P(A^c) \), the probability that no two people among 1, …, \( n \) have the same birthday:

\[- P(A^c) = B_1 \cdot B_2 \cdot \ldots \cdot B_n \]

- What is \( B_2 \)?

\[- B_2 = \frac{364}{365} \text{ because one birthday has been “taken” so far} \]

- \( B_i = \frac{365-i+1}{365} \)
“The Birthday Paradox”

- Let’s try to calculate $P(A^c)$, the probability that no two people among 1, ..., $n$ have the same birthday:

  - $P(A^c) = B_1 \cdot B_2 \cdot ... \cdot B_n$

  - $P(A^c) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{365-n+1}{365}$

    \[ = \left( \frac{365!}{365^n(365-n)!} \right) \]

What is $P(A^c)$ when $n = 1$?

What is $P(A^c)$ when $n = 23$?
Closing the Loop

• “No birthday gets hit twice …” reminds you of …

• … an injection !

• Probability that a random function from \{1, 2, 3, \ldots, n\} to \{1, 2, 3, 4, \ldots, 364, 365\} is an injection is …

\[
\frac{(365! \ (365-n)!)}{365^n}
\]
Mean or Expected Value

• Mean value = expected value of one sample from a random variable
  – I.e., “average”

• If the random variable $X$ can take values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$, then the mean or expected value of $X$, written $E[X]$, is

$$E[X] = \sum_{k=1}^{n} (p_k x_k)$$
Expected Value

• Suppose that the random variable X takes on values 
\( \{x_1=1, x_2=2, x_3=3, x_4=4, x_5=5\} \) with probabilities

\[
\begin{align*}
p_1 &= P(X = x_1) = .05 \\
p_2 &= P(X = x_2) = .05 \\
p_3 &= P(X = x_3) = .2 \\
p_4 &= P(X = x_4) = .3 \\
p_5 &= P(X = x_5) = .4 
\end{align*}
\]

Then \( E[X] = \sum_{k=1}^{5} (p_k x_k) = \)

\[
= (1)(.05) + (2)(.05) + (3)(.2) + (4)(.3) + (5)(.4) \\
= 3.95
\]
Variance

• Definition: The variance of a random variable is

\[ \sigma_x^2 = \text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \]

• → Expected value of \( X^2 \), minus (expected value of \( X \)), squared

• Variance how much \( X \) is likely to vary through its distribution.
Variance Example

• Again: X takes values \{x_1=1, x_2=2, x_3=3, x_4=4, x_5=5\} with probabilities:

  \[ p_1 = P(X = x_1) = .05 \]
  \[ p_2 = P(X = x_2) = .05 \]
  \[ p_3 = P(X = x_3) = .2 \]
  \[ p_4 = P(X = x_4) = .3 \]
  \[ p_5 = P(X = x_5) = .4 \]

\[
E[X^2] = \sum_{k=1}^{5} (p_k x_k^2)
\]
\[
= (1)(.05) + (4)(.05) + (9)(.2) + (16)(.3) + (25)(.4)
\]
\[
= 16.85
\]

\[
E[X] = 3.95 \text{ from before}
\]

\[
Var(X) = E(X^2) - (E(X))^2 = 16.85 - (3.95)^2 = 1.2475
\]
Graphs

Gene-gene interactions affecting phenotype in yeast
Graphs

Airline routes

Control/data flow graph
Graph Representation

- Graph $G = (V,E)$
  - Vertex set $V = V(G)$ $V = \{1, 2, 3, 4, 5\}$
  - Edge set $E = E(G)$ (connecting pairs of vertices)
    $E = \{(1,3), (1,4), (2,3), (2,4), (3,5), (4,5)\}$

- There is a HW 4 question is on *random graphs*
- Class $G(n,p)$ of random graphs:
  - $|V| = n$ vertices
  - Every edge $e = (v_i,v_j)$ exists independently, with $\text{Prob}(e) = p$
Graphs

Internet topology
(random growth models have been proposed)
Reading – Keywords!

- Random variables
- Distribution function of r.v.
- Expectation
- Variance
- Linearity of expectation
- Binomial distribution, Hypergeometric distribution
- Poisson distribution
- Normal distribution
- Joint distribution
- Marginal distribution
- Independence
Fun (from Lecture 6)

• If two sets can be put into 1-1 correspondence, their cardinalities are equal.

• “Cardinality” of infinite sets: very interesting!

“Elvis’s Hotel”: although it’s always crowded, you still can find some room …

Elvis’s Hotel has Rooms 1, 2, 3, … Every room is occupied by a guest.

− One person arrives. Can he be accommodated?

− Five people arrive. Can they be accommodated?

− |N| people arrive. Can they be accommodated?
Fun (from Lecture 6)

• If two sets can be put into 1-1 correspondence, their cardinalities are equal.

• Which set has larger cardinality:
  – R (reals) or (0,1) ? SAME
  – N (counting numbers) or $N \times N = Q^+$ (positive rationals) ?
  – N (counting numbers) or $R^+$ (positive reals) ?
Problems 8

• P8.1 Prove that no matter how you place 33 rooks on an 8 x 8 chessboard, there will always be five of them that are mutually non-attacking.

• P8.2 Are you able to choose a set of 1983 (distinct) non-negative integers, all less than 100000, such that no three are in arithmetic progression? (Hint 1: What happens if you greedily construct such a set, starting with 0? Hint 2: What does your set look like in ternary (base 3) notation?)

• P8.3 Start with two points on a line labeled 0, 1 in that order. In one move you may add or delete two neighboring points (0,0) or (1,1). Your goal is to reach a single pair of points labeled (1,0) in that order. Can you achieve this goal?
EXTRA
• A man is dealt 4 spade cards from an ordinary deck of 52 cards. He is then given 2 more cards. Let x be the probability that both of these cards are of the same suit. Which is true?

• (a) \(0.2 < x \leq 0.3\)
• (b) \(0 < x \leq 0.1\)
• (c) \(0.1 < x \leq 0.2\)
• (d) \(0.3 < x \leq 0.4\)
• (e) \(0.4 < x \leq 0.5\)
Probability Warmups

• You roll two fair dice. What is the probability that the greatest common divisor of the two numbers rolled is 1?

(a) 12/36 (b) 15/36 (c) 17/36 (d) 19/36 (e) 23/36
Probability Warmups

• You are given a square with side $s$, and completely contained inside the square, a circle of radius $r < s$. A point is selected uniformly at random from the square. What is the probability that the selected point lies inside the circle?
Probability Warmups

• You play a coin-tossing game in which you continue to flip a fair coin until it first turns up heads. Consider the potential third flip in this game.

• Given that the game reaches a third flip, what is a probability that the coin turns up heads on the third flip?

• What is the probability that there is a third flip in the game, and that flip turns up heads?
Birthday Paradox / Problem

• Attempting to calculate $P(\text{not } A)$, the probability that no two people 1, …, $n$ have the same birthday:

  – $P(\text{not } A) = \left(\frac{365!}{365^n(365-n)!}\right)$

  – Let's try out $n = 1$:

  – $P(\text{not } A \text{ for } n = 1) = \left(\frac{365!}{365(364)!}\right) = \left(\frac{365}{365}\right) = 1$

In other words, there is a 100% chance that 1 person alone in a room doesn't have the same birthday as anyone else in the room!
Birthday Paradox / Problem

• Attempting to calculate $P(\text{not } A)$, the probability that no two people 1, ..., $n$ have the same birthday:

  - $P(\text{not } A) = \left( \frac{365!}{365^n(365-n)!} \right)$

  - Let's try out $n = 23$:

  - $P(\text{not } A \text{ for } n = 23) = \left( \frac{365!}{365^{23}342!} \right) = .4927$

  - So $P(A) = 1 - .4927 = .5073$

  - In other words, with 23 people in a room, chances are around 50-50 that two of them have the same birthday!