Lecture 6 Notes

- **Goals:** CL Sections 3, 4; FN Section 1
  - Probability
  - Functions

- **Try the new generated questions on WeBWorK!**
  - Feedback?

- **Weekend OHs for TAs/tutors will be posted**

- **MT review:** Wednesday 4/30 6:30pm, location TBD
  - MT is two weeks from today
  - Practice MT questions posted by ~4/25-26 (Fri-Sat)
Review: n Objects into k Marked Boxes

- Given n indistinguishable objects and k marked boxes, there are \( C(n + k - 1, n) = C(n + k - 1, k - 1) \) ways to distribute the objects among the boxes.
  = Order Form Method

- Distributing n distinct objects into k marked boxes is equivalent to lining up the objects in a row and stamping one of the k different box names on each object. There are \( k \cdot k \cdot \ldots \cdot k = k^n \) ways to do this.

- Given n distinct objects and k marked boxes, for \( m_1 + m_2 + \ldots + m_k = n \) there are \( (n; m_1,m_2,\ldots,m_k) = n! / (m_1!m_2!\ldots\cdot m_k!) \) ways to put \( m_i \) objects into the \( i^{th} \) box for all \( i = 1, \ldots, k \).
  = Multinomial  (CL, Example 18)
Review: Probability

- Sample space $U$ = universal set of possible outcomes
- Event $E$ = collection of possible outcomes, $E \subseteq U$

- $\Pr(\text{union of events}) \leq \text{sum of probabilities of the individual events}$
- $\Pr(\text{union of events}) = \text{sum of probabilities of the events when the events are disjoint}$
  - $\Pr(\text{rolling an odd or a prime}) = \frac{4}{6} < \frac{1}{2} + \frac{1}{2}$
  - $\Pr(\text{rolling a 6 or a 2}) = \frac{1}{3} = \frac{1}{6} + \frac{1}{6}$

- $\Pr(\text{A and B}) = \Pr(A) \cdot \Pr(B)$ when $A, B$ independent

- $\Pr(A) = 1 - \Pr(A^c)$

$\Pr(\text{event}) = 1 - \Pr(\text{event’s complement})$
Tossing a Fair Coin

- Probability of 3 heads in 5 tosses of a fair coin

- $2^5 = 32$ possible outcomes for 5 coin tosses:
  $U = \{HHHHH, HHHHT, HHHTH, HHHTT, \ldots, TTTTT\}$

- How many ways to obtain 3 heads?
  - $C(5,3) = 10$ ways to place the 3 H’s in a sequence of 5 tosses, e.g., HTTHH or THHTH, etc.

- $Pr(3 \text{ heads in 5 tosses}) = \frac{10}{32} = \frac{5}{16}$. 
Tossing a Fair Coin

- Probability of 3 or more heads in 5 tosses of fair coin

\[
P(3 \text{ or more heads in 5 tosses}) = \frac{\binom{5}{3}}{32} + \frac{\binom{5}{4}}{32} + \frac{\binom{5}{5}}{32} = \frac{10 + 5 + 1}{32} = \frac{1}{2}.
\]

Note: \(P(3 \text{ or more H's}) = P(3 \text{ or more T's}) = P(2 \text{ or fewer H's}) = 1/2\)

no calculation, just apply symmetry…!
Tossing a Biased Coin

• Probability of 3 H’s in 5 tosses of a biased coin where Pr(H) = 2/3 and Pr(T) = 1/3

• Pr(HHHTT) = (2/3)(2/3)(2/3)(1/3)(1/3) = 8/243

• But I need to add Pr(HHTHT), Pr(HHTTH), etc. → C(5,3) ways in which 3 out of 5 tosses can return heads

• C(5,3) * (8/243) = 80/243
Spinning a Roulette Wheel

• Probability of exactly 3 A’s, 2 B’s and 1 C in 6 spins of a roulette wheel with \( \Pr(A) = \Pr(B) = \Pr(C) = 1/3 \)

\[
\Pr = \binom{6}{3,2,1} \cdot (1/3)^6 \\
= 6! / (3! \cdot 2! \cdot 1!) \cdot (1/3)^6 \\
= 60 / 729 = 20 / 243
\]

• Probability of exactly 3 A’s, 2 B’s and 1 C in 6 spins of a roulette wheel with \( \Pr(A) = 1/12, \Pr(B) = 2/3, \Pr(C) = 1/4 \)

\[
\Pr = \binom{6}{3,2,1} \cdot (1/12)^3 \cdot (2/3)^2 \cdot (1/4)^1
\]
Drawing Three to a Flush  (hint: bad idea)

• You’re playing poker with a standard deck of cards. You’re dealt two clubs and you hope to get a flush, i.e. three more clubs, on the next three cards. What is the probability that this happens?

• #ways to draw three more cards =

• #ways to draw three clubs =

11 remaining clubs among the 50 cards that you haven’t seen yet \( \Rightarrow \) \( \text{Pr} = \frac{C(11,3)}{C(50,3)} \) chances are not good
Socks

• You have 12 red socks and 5 blue socks in your sock drawer (and, no other socks). You choose two socks at random. What is the probability that you have chosen a matching pair?
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Answer. There are \( C(17,2) = 136 \) different pairs of socks that you could choose. Of these, \( C(12,2) = 66 \) pairs are both red, and \( C(5,2) = 10 \) pairs are both blue. \( Pr = \frac{76}{136} = \frac{19}{34} \). Just to check, we can verify that there are 12 * 5 = 60 pairs that are mismatched.
Example With Case Analysis …

- Fifteen balls are numbered 1 through 15. Four of the balls are selected at random without replacement. What is the probability that the sum of the numbers on the selected four balls is odd?
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- Fifteen balls are numbered 1 through 15. Four of the balls are selected at random without replacement. What is the probability that the sum of the numbers on the selected four balls is odd?

**Answer.** The sum can be odd in either of two ways: (1) if one ball is odd and the other three are even; or (2) if three are odd and one is even. Of the 15 numbered balls, 8 of them are odd and 7 are even. Case 1, therefore, has probability $\frac{8}{15}$ of choosing one odd ball and $(\frac{7}{14})(\frac{6}{13})(\frac{5}{12}) = \frac{5}{52}$ for choosing three even balls. Combined, that probability is $\frac{2}{39}$. However, this can happen in four different ways, as the odd ball can be chosen first, second, third or fourth; hence, the total probability for case 1 is $4 \times \frac{2}{39} = \frac{8}{39}$. Similarly, case 2 has probability $(\frac{8}{15})(\frac{7}{14})(\frac{6}{13})(\frac{7}{12}) = \frac{14}{195}$, and there are four ways for that to happen, giving $\frac{56}{195}$.

**The combined probability is** $\frac{8}{39} + \frac{56}{195} = (40 + 56)/195 = \frac{96}{195} = \frac{32}{65}$. 
Functions

Unit FN, Definition 1 (Function).

- If $A$ and $B$ are sets, a function from $A$ to $B$ is a rule that tells us how to find a unique $b \in B$ for each $a \in A$. We write $f: A \rightarrow B$ to indicate that $f$ is a function from $A$ to $B$.

- We call the set $A$ the domain of $f$ and the set $B$ the range (or, codomain) of $f$. To specify a function completely you must give its domain, range and rule.
Functions

• Is the following a complete and correct description of a function?

\[ f : \{1,2,3\} \rightarrow \{1,2,3,4\} , \quad f(x) = x + 1 \]

• A. Yes
• B. No
• C. Unclear
Functions

• Is the following a complete and correct description of a function?

\[ g : \{1,2,3\} \rightarrow \{1,2,3,4\} , \quad g(x) = x - 1 \]

• A. Yes
• B. No
• C. Unclear
Functions

• Is the following a complete and correct description of a function?

\[ h : \{a,b,c\} \rightarrow \{1,2,3,4\} , \quad h(x) = x + 1 \]

• A. Yes
• B. No
• C. Unclear
Functions

• Is the following a complete and correct description of a function?

\[ f : \{1,2,3\} \rightarrow \{1,2,3,4\} , \]

\[
\begin{array}{c|c}
 x & f(x) \\
\hline
 1 & 2 \\
 2 & 3 \\
 3 & 4 \\
\end{array}
\]

• A. Yes
• B. No
• C. Unclear
Is the following a complete and correct description of a function?

\[ f : \{1,2,3\} \rightarrow \{1,2,3,4\}, \]

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

A. Yes  
B. No  
C. Unclear
Properties of Functions

• Is the function $f$ injective, i.e., 1-to-1?

$f : \{1,2,3\} \rightarrow \{1,2,3,4\} , \quad f(x) = x + 1$

• A. Yes
• B. No
• C. Unclear

Note: *injective* == “nobody (in the range) gets hit twice”
Properties of Functions

• Is the function $f$ surjective, i.e., onto?

$f : \{1,2,3\} \rightarrow \{1,2,3,4\}$, \hspace{1cm} f(x) = x + 1

• A. Yes
• B. No
• C. Unclear

Note: surjective == “everybody (in the range) gets hit”
Properties of Functions

• Is the function $f$ *bijective*, i.e., *1-1 and onto*?

\[ f : \{1,2,3\} \rightarrow \{1,2,3,4\}, \quad f(x) = x + 1 \]

• A. Yes
• B. No
• C. Unclear

Note: if $f$ is a bijection, its inverse $f^{-1}$ is well-defined

Note: if $f$ is a bijection, then the cardinalities of its domain and its range are equal
Properties of Functions

- Consider:
  - $\mathbb{R} = \text{real numbers}$
  - $\mathbb{R}_{\geq 0} = \text{non-negative real numbers}$
  - $(0,1) = \text{open interval } \{x \mid 0 < x < 1\}$
  - $\Sigma = \text{letter of the English alphabet } \{a, b, c, \ldots, y, z\}$
  - $L = \text{all English words} \{\text{cat, dog, hat, function, } \ldots\}$
  - $\Sigma^* = \text{all lists of 0 or more English letters} \{\varepsilon, a, b, aa, ab, \ldots\}$

Is $f : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}, \ f(x) = x^2$ a function? 1-1? onto?

Is $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}, \ f(x) = -x$ a function? 1-1? onto?
Properties of Functions

• Consider:
  – \( \mathbb{R} \) = real numbers
  – \( \mathbb{R} \geq 0 \) = non-negative real numbers
  – \( (0,1) \) = open interval \{x | 0 < x < 1\}
  – \( \Sigma = \) letter of the English alphabet \{a, b, c, \ldots, y, z\}
  – \( L = \) all English words \{cat, dog, hat, function, \ldots\}
  – \( \Sigma^* = \) all lists of 0 or more English letters \{\varepsilon, a, b, aa, ab, \ldots\}

Is \( f : L \to \Sigma \)
  \( f(\omega \in L) = \) first letter of \( \omega \) a function?
  1-1? onto?

Is \( f : \Sigma \to L \)
  \( f(\sigma \in \Sigma) = \) all words \( \omega \in L \) beginning with \( \sigma \)
a function? 1-1? onto?

Is \( f : L \to \Sigma^* \)
  \( f(\omega \in L) = \) the reverse of \( \omega \) a function?
  1-1? onto?
Fun

• If two sets can be put into 1-1 correspondence, their cardinalities are equal.

• Observation: “cardinality” becomes really interesting when sets are infinite …

  “Elvis’s Hotel”: although it’s always crowded, you still can find some room …

• So … (let’s call this Problem 6.5)

Which set has larger cardinality:
  – R (reals) or (0,1) ?
  – N (counting numbers) or N × N = Q⁺ (positive rationals) ?
  – N (counting numbers) or R (reals) ?
Problems 6: Pigeonhole, Ramsey

• P6.1 Nine points are in a unit square. Prove that there is a triangle formed by three of the points whose area is not more than 1/8.

• P6.2 Six persons are in a room. Every pair of persons is either friends or enemies. Prove that there must be a subset of three persons (out of the six) who are mutual friends or mutual enemies.
  — Put another way: Imagine a graph with six vertices, and all C(6,2) possible edges. Every edge is colored either red or blue. Prove that the graph must have either a red triangle or a blue triangle.

• P6.3 51 numbers are chosen from the set {1, ..., 100}. Prove that two of the chosen numbers are consecutive.

• P6.4 10 numbers are chosen from the set {1, ..., 100}. Prove that we can find two disjoint non-empty subsets of the chosen numbers such that the subsets each have the same sum of elements.
EXTRA SLIDES (Problem Backlog)

One CSE 21 poker chip to the first correct written (paper or email) solution for each problem that hasn’t already had its solution posted: redeemable for something useful at end of quarter (coffee, points, …)
Problems 1: “Parity, Pigeonhole, Invariants”

• P1.1 You have 7 glasses, all right-side up. In any single move, you can turn over exactly 4 glasses. Is there a sequence of moves that end up with all 7 glasses upside-down? Explain.

• P1.2 Prove that if five points lie in a 2 x 2 square region, some two of them must be no more than sqrt(2) distance apart.

• P1.3 Given any small positive real ε > 0, prove that there is some integer n such that nπ is within ε of an integer value.

• P1.4 Given the set of integers S = {1, 2, …, 200}. Prove that no matter how you choose 101 numbers from S, there will be two chosen numbers such that one evenly divides the other.
Problems 1: “Parity, Pigeonhole, Invariants”

• P1.5 Three frogs are initially located in the infinite plane at points (1,1), (1,0) and (0,0). When one frog jumps over another frog, its position is reflected about the other frog’s position. E.g., if the frog at (0,0) jumps over the frog at (1,0), it lands at (2,0). Can one of the three frogs ever get to (1, 1)? Explain.

• P1.6 At a big international conference, many handshakes are exchanged. We call a person an *odd person* if she has exchanged an odd number of handshakes. Otherwise, she is called an *even person*. Explain why, at any moment, there is an even number of odd persons.
Problems 2: (more) Pigeonhole, Invariants

- P2.1 A dragon has 100 heads. A knight can cut off 15, 17, 20, or 5 heads with one blow of his sword. In each of these cases, 24, 2, 14, or 17 new heads, respectively, will grow back. If all heads are cut off, the dragon dies. Can the dragon ever die? Explain.

- P2.2 There are N persons in a room. Prove that among them there are two persons who have the same number of acquaintances in the room.

- P2.3 Every point in the plane is colored either red or blue. Prove that somewhere in the plane, there are three vertices of an equilateral triangle which are all the same color.

- P2.4 Inside a room of area 5, you place 9 rugs, each of area 1 and having an arbitrary shape. Explain why there must be two rugs which overlap by at least 1/9. Hint: $(1 + 2 + \ldots + 9)/9 = 5$
Problems 3: Induction, “Extremal” Arguments

• P3.1 What is the error in the following induction proof?

Claim: All horses are of the same color.

Proof by mathematical induction (on the size of a group of horses).

Base case (n = 1). Consider a group of horses of size 1, i.e., a single horse. All the horses in this group are clearly of the same color.

Induction hypothesis. Any group of k horses is of the same color.

Induction step. Consider a group of k+1 horses. Take out one of the horses, say H₁, leaving k horses H₂, ..., Hₖ₊₁. By the induction hypothesis, all of these k horses are of the same color C. Now take a different horse, say H₃, leaving k horses H₁, H₂, H₄, ..., Hₖ₊₁. Again by the induction hypothesis, all of these horses are of the same color, C’. But since the groups overlap, the colors C and C’ must be the same, and hence all k+1 horses are of the same color.

Since we have proved that a group of horses of any size is of a single color, and there exist only a finite number of horses on the planet, we have that all horses must be of the same color.
Problems 3: Induction, “Extremal” Arguments

• P3.2 Into at most how many parts is a plane cut by n lines? (Give your answer as an expression in n, and justify.)

• P3.3 There are n points given in the plane. Any three of the points form a triangle of area $\leq 1$. Explain why all n points must lie in a triangle of area $\leq 4$.

• P3.4 You are given 2n points are given in the Euclidean plane, with no three of these points collinear. Exactly n of these points are farms $F = \{F_1, F_2, \ldots, F_n\}$. The other n points are wells $W = \{W_1, W_2, \ldots, W_n\}$. Explain why it is possible to build n straight-line roads that connect each farm to exactly one well, and each well to exactly one farm (notice that the roads define a 1-to-1 correspondence between the sets F and W), such that none of the roads intersect.
Problems 4: Pigeonhole Again

• P4.1 Six points are placed in a 3 x 4 rectangle. Explain why at least two of the points are within distance $\sqrt{5}$ of each other.
Problems 5: A Lower Bound for Sorting

- P5.1 Can you prove by induction that a binary tree of height $h$ has at most $2^h$ leaves?

- P5.2 Can you prove that $\log_2(n!) \leq n \cdot \log_2 n$?

- P5.3 Can you prove that $(n/2) \cdot \log_2(n/2) \leq \log_2(n!)$?

- Comment: If you can solve P5.1 – P5.3, then, as mentioned last time, you will have shown that any algorithm that sorts by comparisons must have at least $n \log n$ worst-case time complexity!!!
  - Recall: a comparison tree that sorts $n$ numbers must have at least $n!$ leaves (== number of possible outcomes)