Anne takes a true-false test with 6 questions. She gets +1 point for each correct answer, -1 point for each wrong answer, and 0 points for each answer left blank. In how many ways can Anne get a score of 4 on the exam?

- A: the sum of the digits of the answer is 4. (E.g., 4 or 13 or 220 or …)
- B: the sum of the digits of the answer is 3.
- C. the sum of the digits of the answer is 2.
- D. the sum of the digits of the answer is 1.
- E. none of the above.

4 points = 4 correct, 2 blank

\[ C(6,2) = 15 \]

\[ 5 \text{ correct, 1 wrong} \]

\[ C(6,1) = 6 \]
Warmup 1

• Anne takes a true-false test with 6 questions. She gets +1 point for each correct answer, -1 point for each wrong answer, and 0 points for each answer left blank. In how many ways can Anne get a score of 4 on the exam?
  – A: the sum of the digits of the answer is 4. (E.g., 4 or 13 or 31 or …)
  – B: the sum of the digits of the answer is 3. (21 ways)
  – C. the sum of the digits of the answer is 2.
  – D. the sum of the digits of the answer is 1.
  – E. none of the above.

Answer by Case Analysis. If Anne scores 4 points by getting 4 correct answers and 2 blanks, then she can do this in C(6,4) = C(6,2) = 15 ways. If Anne scores 4 points by getting 5 correct answers and 1 wrong answer, then she can do this in C(6,5) = C(6,1) = 6 ways. Total = 15 + 6 = 21 ways.
100 people in Whosville watch baseball. 63 of them watch the Alphas. 45 watch the Betas. 30 watch the Gammas. Furthermore, 21 watch both (but not necessarily exclusively) the Alphas and the Betas, and 15 watch both the Betas and the Gammas. 9 diehard fans watch all three teams. How many fans watch both the Alphas and the Gammas?

- A: the sum of the digits of the answer is 4. (E.g., 4 or 13 or 31 or …)
- B: the sum of the digits of the answer is 3.
- C. the sum of the digits of the answer is 2.
- D. the sum of the digits of the answer is 1.
- E. none of the above.

\[
\begin{align*}
    x + y &= 42 \\
    y + z &= 15 \\
    x + y + z &= 55
\end{align*}
\]
Warmup 2

100 people in Whosville watch baseball. 63 of them watch the Alphas. 45 watch the Betas. 30 watch the Gammas. Furthermore, 21 watch both (but not necessarily exclusively) the Alphas and the Betas, and 15 watch both the Betas and the Gammas. 9 diehard fans watch all three teams. How many fans watch both the Alphas and the Gammas?

- C. the sum of the digits of the answer is 2.

Answer. Drawing out the Venn diagram, we can see from the given facts that \(|(A \cap B) \setminus G| = 12,\)
\(|(B \cap G) \setminus A| = 6,\) and \(|B \setminus (A \cup G)| = 18.\) There are three remaining pieces of the puzzle: \(|A \setminus (B \cup G)| =\)
the number of fans who watch only the Alphas (and neither the Betas nor the Gammas), \(|G \setminus (A \cup B)| =\)
the number of fans who watch only the Gammas (and neither the Alphas nor the Betas), and that \|(A \cap G) \setminus B| =\)
the number of fans who watch both the Alphas and the Gammas, but not the Betas. Since 63 watch the Alphas, we have \(|A \setminus (B \cup G)| + |(A \cap G) \setminus B| = 63 - 21 = 42.\) Since
30 watch the Gammas, we have \|(A \cap G) \setminus B| + |G \setminus (A \cup B)| = 30 - 15 = 15.\) Since there are 100 total fans, we have \(|A \setminus (B \cup G)| + |(A \cap G) \setminus B| + |G \setminus (A \cup B)| = 100 - 45 = 55.\) Therefore, \|(A \cap G) \setminus B| = 2
\Rightarrow 9 + 2 = 11 fans watch both the Alphas and the Gammas.
Lecture 5 Notes

• Goal for this week: CL Sections 3, 4
  – Counting problem types (identical, distinct balls/bins)
  – Binomial and Multinomial Coefficients
  – Probability!

• I will start posting solutions to old problems after, say, 2 weeks
  – See, for example, the last few slides in this lecture’s handout
  – This being said, please do try the Px.y problems on your own!

• Soon: generated drill questions on WeBWorK!
  – Order form
  – Rearranging letters
  – … (any requests?)
Putting n Objects into k Marked Boxes

- Given n indistinguishable objects and k marked boxes, there are \( C(n + k - 1, n) = C(n + k - 1, k - 1) \) ways to distribute the objects among the boxes.

= Order Form Method
How many ways are there to pick a collection of 10 balls from a pile of red balls, blue balls, and purple balls if at least 5 red balls must be picked?

\[ C(7,2) = C(7,5) \]
How many ways are there to pick a collection of 10 balls from a pile of red balls, blue balls, and purple balls if at least 5 red balls must be picked?

ANSWER: 21 ways. Simplify the problem by first picking the 5 required red balls. Then, pick the remaining 5 balls arbitrarily (possibly including more red balls). Picking 5 balls from 3 colors (with replacement) can be done using the Order Form Method: \( \binom{7}{2} = 21 \) ways.
... and Counting the Complement

- How many ways are there to pick a collection of 10 balls from a pile of red balls, blue balls, and purple balls if at most 5 red balls must be picked?

\[
\binom{12}{2} - \binom{7}{2} - \binom{6}{2}
\]

\[\text{\# ways to pick 10 balls}\]

\[\text{\# ways to choose at least 6 red balls}\]

\[\text{\# ways to choose at least 5 red balls}\]
... and Counting the Complement

- How many ways are there to pick a collection of 10 balls from a pile of red balls, blue balls, and purple balls if at most 5 red balls must be picked?

**ANSWER:** Count the complement! There are $C(12,2) = 66$ ways to pick 10 balls without restriction from the three colors. If you pick 6 red balls, then there are $C(6,2) = 15$ ways to pick 4 remaining balls arbitrarily (possibly including more red balls). The number of ways to pick 10 balls without choosing more than 5 red balls is $66 - 15 = 51$. 

Putting n Objects into k Marked Boxes

• Distributing n distinct objects into k marked boxes is equivalent to lining up the objects in a row and stamping one of the k different box names on each object. There are $k \times k \times \ldots \times k = k^n$ ways to do this.

• In how many ways can I assign 100 students to the Left, Center, and Right sections of the classroom?
Putting \( n \) Objects into \( k \) Marked Boxes

- Distributing \( n \) distinct objects into \( k \) marked boxes is equivalent to lining up the objects in a row and stamping one of the \( k \) different box names on each object. There are \( k \cdot k \cdot \ldots \cdot k = k^n \) ways to do this.

- In how many ways can I assign 100 students to the Left, Center, and Right sections of the classroom?

  \[
  \left\{ \text{L, C, R} \right\}^{100}
  \]

  Each student: 3 possible assignments \( \rightarrow 3^{100} \) ways
Putting $n$ Objects into $k$ Marked Boxes

- Given $n$ distinct objects and $k$ marked boxes, for $m_1 + m_2 + \ldots + m_k = n$ there are
  
  \[ \binom{n}{m_1,m_2,\ldots,m_k} = \frac{n!}{m_1!m_2!\ldots m_k!} \]

  ways to put $m_i$ objects into the $i^{th}$ box for all $i = 1, \ldots, k$.

  = Multinomial (see Example 18 in text)

- In how many ways can I assign 100 students to the Left, Center, and Right sections of the classroom with 40 on the Left, 40 in the Center, and 20 on the Right?

  \[ \begin{align*}
  n &= 100 \\
  m_1 &= 40 \\
  m_2 &= 40 \\
  m_3 &= 20 \\
  \binom{100}{40,40,20} &= \frac{100!}{40!40!20!} \\
  \end{align*} \]
Putting n Objects into k Marked Boxes

• Given n distinct objects and k marked boxes, for \( m_1 + m_2 + \ldots + m_k = n \) there are
\[
\binom{n}{m_1,m_2,\ldots,m_k} = \frac{n!}{(m_1!) (m_2!) \ldots (m_k!)}
\]
ways to put \( m_i \) objects into the \( i^{\text{th}} \) box for all \( i = 1, \ldots, k \).

= Multinomial  (see Example 18 in text)

• In how many ways can I assign 100 students to the Left, Center, and Right sections of the classroom with 40 on the Left, 40 in the Center, and 20 on the Right?

\[
\binom{100}{40} \times \binom{60}{40} \times \binom{20}{20}
\]

\[
= \frac{100!}{(40! 60!)} \times \frac{60!}{(40! 20!)} \times \frac{20!}{(20! 0!)}
\]

\[
= \frac{100!}{(40! 40! 20!)}
\]

ways
Putting \( n \) Objects into \( k \) Marked Boxes

- Given \( n \) distinct objects and \( k \) marked boxes, for \( m_1 + m_2 + \ldots + m_k = n \) there are
  \[ (n; m_1, m_2, \ldots, m_k) = \frac{n!}{m_1!m_2! \ldots m_k!} \]
  ways to put \( m_i \) objects into the \( i \)th box for all \( i = 1, \ldots, k \).
  = Multinomial (see Example 18 in text)

- In how many distinct ways can I rearrange the letters of the word GOOGLE?

\[ \frac{6!}{2!2!} = \frac{720}{4} = 180 \]
Putting n Objects into k Marked Boxes

- Given n distinct objects and k marked boxes, for \( m_1 + m_2 + \ldots + m_k = n \) there are
  \( (n; m_1,m_2,...,m_k) = \frac{n!}{(m_1!m_2!\ldots m_k!)} \) ways to put
  \( m_i \) objects into the \( i^{th} \) box for all \( i = 1, \ldots, k \).
  = Multinomial  (see Example 18 in text)

- In how many distinct ways can I rearrange the letters of the word GOOGLE?

- The n objects are the positions in the arrangement:
  \( __, __, __, __, __, __ \)

- The k marked boxes are the letters G, O, L, E

- \( (6; 2,2,1,1) = 6! / (2! 2! 1! 1!) = 720 / 4 = 180 \) ways
Binomial Theorem, Binomial Coefficients

• \( C(n,k) = \frac{n!}{(n-k)! \, k!} \)
• \( C(n,k) = C(n, n-k) \)

• Can view as multinomial with \( k = 2 \):
  – \( C(n,k) = (n; k, n-k) = (n; n-k, k) \)

• Binomial Theorem
• \( (1+x)^n = C(n,0)x^0 + C(n,1)x^1 + C(n,2)x^2 + \ldots + C(n,n)x^n \)
• \( (a+b)^n = C(n,0)a^0b^n + C(n,1)a^1b^{n-1} + \ldots + C(n,n)a^nb^0 \)

\[
(1+x)^3 = 1 + 3x + 3x^2 + x^3
\]
Binomial Theorem, Binomial Coefficients

\[(1+x) \times (1+x) \times (1+x) \times (1+x) = (1+x)^4\]

How do I get, e.g., an \(x^2\) term from this product?

2 \(x\)'s, 2 1's in \(C(4,2)\) ways!

\[= C(4,0)x^0 + C(4,1)x^1 + C(4,2)x^2 + C(4,3)x^3 + C(4,4)x^4\]

\[= 1 + 4x^1 + 6x^2 + 4x^3 + x^4\]
Binomial Theorem, Binomial Coefficients

- $C(n,k) = \frac{n!}{(n-k)! \, k!}$
- $C(n,k) = C(n, n-k)$
- $C(n,k) = C(n-1,k) + C(n-1,k-1)$
  - A recurrence! (= recursion)
  - Why is this true?
- Note: $C(n,k)$ = coefficient of $x^k$ in expansion of $(1 + x)^n$
  ..... leads to concept of generating function
Probability

- Sample space \( U \) = universal set of possible outcomes
  - \( \{H, T\} \) // tossing a coin
  - \( \{1, 2, 3, 4, 5, 6\} \) // rolling a six-sided die
  - \( \{A, B, C\} \) // grade in class 😊
  - \( \{HH, HT, TH, TT\} \) // tossing a coin twice

- We say that we choose an element of \( U \) uniformly at random if all elements of \( U \) have equal probability of being chosen.

- An event \( E \) is a collection of possible outcomes, where \( E \subseteq U \).
  - e.g., \( E = \{1, 3, 5\} \) \( \equiv \) roll an odd number

- If outcomes are chosen uniformly at random, the probability of an outcome belonging to an event \( E \) is \( |E| / |U| \).
  - \( E = \{1, 3, 5\} \); \( U = \{1, 2, 3, 4, 5, 6\} \) \( \Rightarrow \) \( \Pr(E) = \frac{1}{2} \)
Probability Facts

• Fact. Probability of the union of events ("or") is the sum of the probabilities of the individual events when the events are disjoint
  – Pr(rolling a 6 or a 2) \rightarrow \frac{1}{3} \quad (= \frac{2}{6})

• Fact: Probability of the union of events is \leq \text{sum of probabilities of individual events (general events)}
  – Pr(rolling an odd number or a prime number)

\{1, 3, 5\} \cup \{2, 3, 5\} = \{1, 2, 3, 5\}

\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}

• Fact: Probability of independent events A and B occurring in their respective trials is \text{Pr(A)} \cdot \text{Pr(B)}.

Pr\left(\text{H on toss 1} \quad \text{and} \quad \text{T on toss 2}\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}

• Fact. \text{Pr(event A)} = 1 – \text{Pr(complement event A}^c\text{)}.
  – Probability of something happening = 1 – Probability of it not happening.
Probability Examples

• Given an urn with 5 red, 5 black balls, you draw two balls without replacement. What is the probability that you have drawn one black and one red ball?

\[
\frac{1}{90} \quad \frac{25}{90} \quad \frac{5}{18} \quad \frac{5}{9}
\]

\[
\frac{25 \text{ (} R_i, B_j \text{) pairs}}{C(10,2) \text{ total pairs}} = \frac{25}{45}
\]

\[
\left( \text{Red first, then Black} \quad \frac{5}{10} \times \frac{5}{9} = \frac{25}{90} \right)
\]

\[
\left( \text{Black first, then Red} \quad \frac{5}{9} \times \frac{5}{10} = \frac{25}{90} \right)
\]
Probability Examples

• Given an urn with 5 red, 5 black balls, you draw two balls without replacement. What is the probability that you have drawn one black and one red ball?

• Assume that all balls are labeled (distinguished from each other)
  – #pairs of balls = C(10,2) = 45
  – #pairs of (black,red) balls = 25
  – 25/45 = 5/9

• Or: No matter what ball you draw first, the probability is 5/9 that the second ball is of the other color
Probability Examples

- Probability of 3 heads in 5 tosses of a fair coin

- There are 32 possible outcomes for 5 coin tosses
  - 2 outcomes for each toss $\rightarrow 2^5 = 32$

- How many ways to obtain 3 heads?
  - $C(5,3) = 10$ ways to place the 3 H’s in a sequence of 5 tosses, e.g., HTTHH or THHTH, etc.

- $Pr(3$ heads in 5 tosses$) = 10/32 = 5/16$. 
• Probability of 3 or more heads in 5 tosses

\[ P(3 \text{ or more heads in 5 tosses}) = P(\text{exactly 3 heads}) + P(\text{exactly 4 heads}) + P(\text{exactly 5 heads}) \]

\[ \frac{\binom{5}{3}}{32} + \frac{\binom{5}{4}}{32} + \frac{\binom{5}{5}}{32} = \frac{10 + 5 + 1}{32} = \frac{1}{2}. \]
Probability Examples

• Probability of 3 heads in 5 tosses of a biased coin where P(heads) = 2/3 and P(tails) = 1/3

• \[ \Pr(\text{HHHTT}) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{8}{243} \]

• But I need to add \( \Pr(\text{HHTHT}) \), \( \Pr(\text{HHTTH}) \), etc. 
  \( \rightarrow \) \( C(5,3) \) ways in which 3 out of 5 tosses can return heads

• \( C(5,3) \times \left( \frac{8}{243} \right) = \frac{80}{243} \)
More!

• Probability of 3 A’s and 2 B’s in 6 spins of a 3-segment roulette wheel. Pr(A) = Pr(B) = Pr(C) = 1/3.

\[
Pr = (6; 3,2,1) \cdot (1/3)^6 \\
= 6! / (3! 2! 1!) \cdot (1/3)^6 \\
= 20 / 243
\]

• Probability of 3 A’s and 2 B’s in 6 spins of a **biased** 3-segment roulette wheel. Pr(A) = 1/12, Pr(B) = 2/3, Pr(C) = 1/4.

\[
Pr = (6; 3,2,1) \cdot (1/12)^3 \cdot (2/3)^2 \cdot (1/4)^1
\]
And More!

• You’re playing poker with a standard deck of cards. You’re dealt two clubs and you hope to get a flush, i.e. three more clubs, on the next three cards. What is the probability that this happens?

• There are 11 remaining clubs among the 50 cards that you haven’t seen yet …

• \( \text{Pr} = \frac{\binom{11}{3}}{\binom{50}{3}} \) chances are not good
More on Probability

• Expectation is like “average” value
  – Will be more formal later

• Notion of a “successful trial” = experiment that results in some defined outcome
  – Getting “H” when tossing a fair coin
  – Getting “6-6” when rolling two fair dice

• Question: What is the expected number of trials before a success is obtained?
  
  \[ = \frac{1}{\Pr(\text{success})} \]
Problems 5: A Lower Bound for Sorting

- P5.1 Can you prove by induction that a binary tree of height $h$ has at most $2^h$ leaves?

- P5.2 Can you prove that $\log_2(n!) \leq n \cdot \log_2 n$?

- P5.3 Can you prove that $(n/2) \cdot \log_2(n/2) \leq \log_2(n!)$?

- **Comment:** If you can solve P5.1 – P5.3, then, as mentioned last time, you will have shown that any algorithm that sorts by comparisons must have at least $n \log n$ worst-case time complexity!!
  
  - Recall: a comparison tree that sorts $n$ numbers must have at least $n!$ leaves (== number of possible outcomes)
Problems 5 (cont.): Extremal Argument

- P5.4 Suppose that N people are standing at (x,y) locations in the plane, and that all pairwise distances (between two people) are unique. If every person simultaneously points at the closest other person, is it possible to have a cycle of more than two people that point to each other (e.g., A -> B -> C -> A would be a cycle of three people)?
EXTRA SLIDES
Selecting Objects

• Suppose that you are given three object types A, B, C.
  – A can be selected 0 or 1 times
  – B can be selected 0, 1 or 2 times
  – C can be selected 0, 1, 2 or 3 times

• In how many ways can you select 5 objects?
Making Change

- Making change with pennies, nickels, dimes
You have 12 red socks and 5 blue socks in your sock drawer. You choose two socks at random. What is the probability that you have chosen a matching pair?
Socks

You have 12 red socks and 5 blue socks in your sock drawer. You choose two socks at random. What is the probability that you have chosen a matching pair?

Answer. There are $\binom{17}{2} = 136$ different pairs of socks that you could choose. Of these, $\binom{12}{2} = 66$ pairs are both red, and $\binom{5}{2} = 10$ pairs are both blue. $\Pr = 76/136 = 19/34$. Just to check, we can verify that there are $12 \times 5 = 60$ pairs that are mismatched.
One Last Example…

• Fifteen balls are numbered 1 through 15. Four balls are then selected at random without replacement. What is the probability that the sum of the numbers on those four balls is odd?
One Last Example…

- Fifteen balls are numbered 1 through 15. Four balls are then selected at random without replacement. What is the probability that the sum of the numbers on those four balls is odd?

**Answer.** The sum can be odd in either of two ways: (1) if one ball is odd and the other three are even; or (2) if three are odd and one is even. Of the 15 numbered balls, 8 of them are odd and 7 are even. Case 1, therefore, has probability $\frac{8}{15}$ of choosing one odd ball and $(\frac{7}{14})(\frac{6}{13})(\frac{5}{12}) = \frac{5}{52}$ for choosing three even balls. Combined, that probability is $\frac{2}{39}$. However, this can happen in four different ways, as the odd ball can be chosen first, second, third or fourth; hence, the total probability for case 1 is $4 \times \frac{2}{39} = \frac{8}{39}$. Similarly, case 2 has probability $(\frac{8}{15})(\frac{7}{14})(\frac{6}{13})(\frac{7}{12}) = \frac{14}{195}$, and there are four ways for that to happen, giving $\frac{56}{195}$.

The combined probability is $\frac{8}{39} + \frac{56}{195} = \frac{40}{195} + \frac{56}{195} = \frac{96}{195} = \frac{32}{65}$. 
An Old Problem, Solved

A Common Problem Type

• How many integer solutions are there to the equation \( x_1 + x_2 + x_3 + x_4 = 12 \), with \( x_i \geq 0 \)?

• How do we apply the Order Form Method?

ANSWER: \( C(15,3) = 455 \). Note that this is like distributing 12 identical objects (i.e., “units”) into 4 distinct boxes \((x_1, x_2, x_3 \text{ and } x_4)\).

• How many solutions are there with \( x_i > 0 \)?

ANSWER: \( C(11,3) = 165 \). This is after putting one “unit” into each of the \( x_i \). There remain 8 “units” to distribute into the 4 distinct boxes.
A Common Problem Type

- How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$, with $x_i \geq 0$?

- How many solutions are there with $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$?

ANSWER: $C(7,3) = 35$. Similarly, after putting 2 units into box $x_1$, 2 units into box $x_2$, and 4 units into box $x_3$, there remain 4 units to distribute into the 4 distinct boxes.
Another Old Problem, Solved

- Bob has three “great books”, four “trashy books”, and two “educational books”. He picks one great book, two trashy books and one educational book to take on the family camping trip. Bob’s mother never lets him read two trashy books in a row. During the trip, Bob read all four books. In how many possible ways could Bob have chosen and read those books?

- Answer: Let’s solve this in two steps: (1) In how many ways can we choose the four books, and (2) Given four books, in how many ways can we order them.
  - Choose the books in \( C(3,1) \times C(4,2) \times C(2,1) = 3 \times 6 \times 2 = 36 \) ways.
  - Given the choice of four books, how many ways to order?
    - 4! = 24 orderings, but this includes orderings that have two trashy books in succession.
    - How many out of the 24 orderings have two trashy books in succession?
      12 = \{Tt__, tT__, _Tt_, _tT_, __Tt, __tT\} \times \{___ = EG, GE\}
    - Note: EG, GE = 2 = 2! ways to order the E, G books.
    - Answer is: 12 allowed orderings
  - Final answer: 36 \times 12 = 432.
And Another Old Problem, Solved

• In how many ways can Bob the dog breeder separate his 10 puppies into a group of 4 and a group of 6, if he has to keep Biter and Nipper, two of the puppies, in separate groups?

• **ANSWER** (don’t look unless you’ve tried the problem!!!) …

112 ways.  Case 1: Biter is in the group of 4.  Then there are $C(8,3) = 56$ ways of picking the other three dogs in the group.  Case 2: Nipper is in the group of 4.  Again there are $C(8,3) = 56$ ways of picking the other three dogs in the group.  $56 + 56 = 112$.  

And Yet Another Old Problem, Solved

• Bob the salesman starts at his home in San Diego, and tours 6 cities (Abilene, Bakersfield, Corona, Denver, Eastville and Frankfort), visiting each city exactly once before returning home. In how many possible ways could Bob make his tour?

There are six choices for the first city in the tour, then five choices for the second city in the tour, etc.. So, there are \( 6! = 720 \) possible tours of form

\[
\text{SD} \rightarrow \_ \rightarrow \_ \rightarrow \_ \rightarrow \_ \rightarrow \_ \rightarrow \_ \rightarrow \text{SD}
\]

Note that \( \text{SD} \rightarrow \text{A} \rightarrow \text{B} \rightarrow \text{C} \rightarrow \text{D} \rightarrow \text{E} \rightarrow \text{F} \rightarrow \text{SD} \) is not the same as \( \text{SD} \rightarrow \text{F} \rightarrow \text{E} \rightarrow \text{D} \rightarrow \text{C} \rightarrow \text{B} \rightarrow \text{A} \rightarrow \text{SD} \).

• In how many possible ways could Bob make his tour if he does not travel directly between San Diego and Frankfort?

The first city in the tour (after SD), and the last city in the tour (before SD), cannot be Frankfort. Frankfort can go in any of 4 positions in the tour. Then, the remaining 5 cities can be ordered in \( 5! \) ways. \( 4 \times 5! = 480 \). This is \( 2/3 \times 6! \) which makes sense: \( 2/6 = 1/3 \) of the positions don’t work for Frankfort in the tour.