Lecture 3 Notes

• Goal for today: CL Section 3
  – Subsets, Binomial Coefficients (poker hands, loop iterations)

• Thanks to Dr. Rubalcaba for giving this lecture 😊
  – (I hope I am in Germany when you see this)

• No office hours tomorrow (Wednesday) 😞
  – (I hope I am flying back from Germany at that point)

• New reader-tutors 😊
  – Kacy Raye Espinoza (krespino@ucsd.edu) and Tracy Nham (tnham@ucsd.edu)
  – Will hold extra OH’s and help sessions, and can also provide 1-1 or small-group help
  – Please contact them to schedule (please cc instructors)
  – We’re adding more to the team as fast as we can
Last Week on One Slide

• Theorem 1: k-lists out of n-set with replacement
• Theorem 4: k-lists out of n-set without replacement
• Theorem 7: k-subsets out of n-set

Recall

• number of k-lists from n-set without repetition: \( \frac{n!}{(n - k)!} \)
• k-list vs. k-subset: each subset is listed in \( k! \) ways

• Theorem 7: number of k-lists of n-set: \( \frac{n!}{[(n - k)! \cdot k!]} \)

• These are combinations of n objects taken k at a time
• Also written \( C(n,k) \) ("n choose k")
Theorems 1, 4, 7

- **Given:** 5-set $S = \{v, w, x, y, z\}$

- **How many 3-lists can be formed from elements of $S$, with repetition?**
  
  [“ordering”: order matters, with repetition]
  
  - Theorem 1 $\implies 5^3 = 125$

- **How many 3-lists can be formed from elements of $S$, without repetition?**
  
  [“ordering”: order matters, without repetition]
  
  - Theorem 4 $\implies \frac{5!}{(5-3)!} = 5 \cdot 4 \cdot 3 = 60$

- **How many 3-sets are subsets of $S$?**
  
  [“selecting”/”choosing”: order doesn’t matter, without repetition]
  
  - Theorem 7 $\implies \frac{5!}{[(5-3)! \cdot 3!]} = 10$
Last Week on One Slide

• Theorem 1: k-lists out of n-set with replacement
• Theorem 4: k-lists out of n-set without replacement
• Theorem 7: k-subsets out of n-set without replacement
• **Rule of Product (Theorem 2):**
  – If structures are constructed by making a sequence of k choices such that the $i^{th}$ choice can be made in $c_i$ ways (independently of previous choices), and each structure arises in exactly one way, then there are $c_1 \cdot c_2 \cdot \ldots \cdot c_k$ possible structures.
• Sets $C_1, \ldots, C_k$ have **Cartesian product** $C_1 \times C_2 \times \ldots \times C_k$ with elements $(x_1, x_2, \ldots, x_k)$ where $x_i \in C_i$, $i = 1, \ldots, k$
• **Lexicographic order** = dictionary order
Last Week on One Slide

• Theorem 1: k-lists out of n-set with replacement
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• **Lexicographic order** = dictionary order
• **Rule of Sum (Theorem 3):** The cardinality of the union of disjoint sets is the sum of cardinalities of those sets
Loop Iterations

• Example: Counting Loop Iterations
  – What is the value of k after the following code has been executed?

\[
k = 0; \\
\text{for } i_1 = 1 \text{ to } n_1 \\
\quad \text{for } i_2 = 1 \text{ to } n_2 \\
\quad \quad \text{for } i_3 = 1 \text{ to } n_3 \\
\quad \quad \quad k = k + 1;
\]

• A: \((n_1)^3\)
• B: \(n_1 \cdot n_2 \cdot n_3\)
• C: \((n_2)^3\)
• D: \(n_1 + n_2 + n_3\)
• E: \(n \cdot (n - 1) / 2\)
Loop Iterations

- Example: Counting Loop Iterations
  - What is the value of k after the following code has been executed?

```plaintext
k = 0;
for i1 = 1 to n1
    k = k + 1;
for i2 = 1 to n2
    k = k + 1;
for i3 = 1 to n3
    k = k + 1;
```

- A: \((n_1)^3\)
- B: \(n_1 \cdot n_2 \cdot n_3\)
- C: \((n_2)^3\)
- D: \(n_1 + n_2 + n_3\)
- E: \(n \cdot (n - 1) / 2\)
Loop Iterations

• Example: Counting Loop Iterations
  – What is the value of k after the following code has been executed?

\[
\begin{align*}
k &= 0; \\
\text{for } i_1 &= 1 \text{ to } n \\
  &\quad \text{for } i_2 = i_1 + 1 \text{ to } n \\
  &\quad \quad k = k + 1;
\end{align*}
\]

- A: \((n_1)^3\)
- B: \(n_1 \cdot n_2 \cdot n_3\)
- C: \((n_2)^3\)
- D: \(n_1 + n_2 + n_3\)
- E: \(n \cdot (n - 1) / 2\)
Poker Hands

- 5-card poker hands from standard 52-card deck

- 52 cards = \{4 suits\} x \{13 ranks\}
  - Suits = ♠, ♥, ♦, ♣
  - Ranks = 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A  [A = highest]

- How many 5-card poker hands?
  - C(52,5) = 2598960

- How many poker hands have 4 of a kind
  == 4 cards of one rank + 1 card of another rank ?
  - A: C(52,4)  B: C(13,1) \cdot C(12,1) \cdot 5!  C: C(13,1) \cdot 48
  - D: C(13,2) \cdot 5!  E: C(13,2) \cdot 48
1-Pair Poker Hands (review this off-line)

• How many 5-card poker hands have exactly one pair?
  – Note: K-K-10-7-3 has one pair and should be counted, but K-K-K-10-7 (three of a kind) or K-K-10-10-7 (two pair) should not be counted)

“Solution A”.

• You can pick the first card in 52 ways.
• You can pick its “pair” in 3 ways.
• This overcounts the #ways we can arrive at the pair by a factor of 2! = 2.
• You can pick each of the next three cards (so as to avoid 3-of-a-kind or two pairs) in 48, 44 and 40 ways, respectively.
• This overcounts the #ways we can arrive at these three cards by a factor of 3! = 6.
• The answer is 52 x 3 x 48 x 44 x 40 / (2 x 6) = 1,098,240.
1-Pair Poker Hands (review this off-line)

• How many 5-card poker hands have exactly one pair?
  – Note: K-K-10-7-3 has one pair and should be counted, but K-K-K-10-7 (three of a kind) or K-K-10-10-7 (two pair) should not be counted

“Solution B”.

• You can pick the four ranks in a one-pair hand in C(13,4) = 715 ways.
• You can pick the rank that has the pair in C(4,1) = 4 ways (choose one rank out of four).
• Within this rank that has the pair, you can pick the actual pair in C(4,2) = 6 ways (e.g., if the rank is Q, then you can pick Q♠ and Q♥).
• For each of the other three ranks, you can pick the actual card in C(4,1) = 4 ways.
• The answer is 715 x 4 x 6 x 4 x 4 x 4 = 1,098,240.
1-Pair Poker Hands (review this off-line)

• How many 5-card poker hands have exactly one pair?
  – Note: K-K-10-7-3 has one pair and should be counted, but K-K-K-10-7 (three of a kind) or K-K-10-10-7 (two pair) should not be counted)

“Solution C”.

• You can pick the one rank containing the pair in \( C(13,1) = 13 \) ways.
• The actual pair of cards can be chosen from this rank in \( C(4,2) = 6 \) ways.
• You can pick the three other ranks in \( C(12,3) = 220 \) ways.
• The actual cards in the other three ranks can be chosen in \( 4 \times 4 \times 4 \) ways.
• The answer is \( 13 \times 6 \times 220 \times 4 \times 4 \times 4 = 1,098,240 \).
A Common Problem Type

• How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$, with $x_i \geq 0$?

• How do we apply the Order Form Method?

• How many solutions are there with $x_i > 0$?
A Common Problem Type

• How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$, with $x_i \geq 0$?

• How do we apply the Order Form Method?

• How many solutions are there with $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$?
Counting Problem Examples from Week 1

Licorice

- How many ways are there to distribute 20 (identical) sticks of red licorice and 15 (identical) sticks of black licorice among five children?
Case analysis

• How many 4-digit campus telephone numbers have one or more repeated digits?

• Cases:
Counting Problem Examples from Week 1

Simplification; Counting the Complement

• How many ways are there to pick a collection of 10 balls from a pile of red balls, blue balls, and purple balls if at least 5 red balls must be picked?

• Simplification:

• What if at most 5 red balls must be picked?
Counting Problem Examples from Week 1

- Bob has three “great books”, four “trashy books”, and two “educational books”. He picks one great book, two trashy books and one educational book to take on the family camping trip. Bob’s mother never lets him read two trashy books in a row. During the trip, Bob read all four books. In how many possible ways could Bob have chosen and read those books?
Counting Problem Examples from Week 1

• In how many ways can Bob the dog breeder separate his 10 puppies into a group of 4 and a group of 6, if he has to keep Biter and Nipper, two of the puppies, in separate groups?
Counting Problem Examples from Week 1

• Bob the salesman starts at his home in San Diego, and tours 6 cities (Abilene, Bakersfield, Corona, Denver, Eastville and Frankfort), visiting each city exactly once before returning home. In how many possible ways could Bob make his tour?

• In how many possible ways could Bob make his tour if he does not travel directly between San Diego and Frankfort?
Problems 3: Induction, “Extremal” Arguments

• P3.1 What is the error in the following induction proof?

Claim: All horses are of the same color.
Proof by mathematical induction (on the size of a group of horses).
Base case (n = 1). Consider a group of horses of size 1, i.e., a single horse. All the horses in this group are clearly of the same color.

Induction hypothesis. Any group of k horses is of the same color.
Induction step. Consider a group of k+1 horses. Take out one of the horses, say H₁, leaving k horses H₂, ..., Hₖ₊₁. By the induction hypothesis, all of these k horses are of the same color C. Now take a different horse, say H₃, leaving k horses H₁, H₂, H₄, ..., Hₖ₊₁. Again by the induction hypothesis, all of these horses are of the same color, C’. But since the groups overlap, the colors C and C’ must be the same, and hence all k+1 horses are of the same color.

Since we have proved that a group of horses of any size is of a single color, and there exist only a finite number of horses on the planet, we have that all horses must be of the same color.
Problems 3: Induction, “Extremal” Arguments

• P3.2 Into at most how many parts is a plane cut by n lines? (Give your answer as an expression in n, and justify.)

• P3.3 There are n points given in the plane. Any three of the points form a triangle of area $\leq 1$. Explain why all n points must lie in a triangle of area $\leq 4$.

• P3.4 You are given 2n points are given in the Euclidean plane, with no three of these points collinear. Exactly n of these points are farms $F = \{F_1, F_2, \ldots, F_n\}$. The other n points are wells $W = \{W_1, W_2, \ldots, W_n\}$. Explain why it is possible to build n straight-line roads that connect each farm to exactly one well, and each well to exactly one farm (notice that the roads define a 1-to-1 correspondence between the sets F and W), such that none of the roads intersect.