Lecture 2 Notes

• Goal for today: Finish CL Sections 1, 2
• Note: I don’t use chalk because I’m very allergic to chalk dust! 😞
• About me: http://vlsicad.ucsd.edu/~abk/
• “Problems”: Px.y is the yth problem from Lecture x
  – Again: These are exercises in creative problem-solving
  – Think of them as practice Google interview questions 😄
  – They exercise specific problem-solving techniques: parity, symmetry, pigeonhole principle, extremal principle, invariants, induction, …

(potential replacement for Threes, 2048, Flappy Bird, …)
Review

- **Set**: collection of distinct objects where order does not matter

- **List** (or **string**): _ordered_ collection
  - (Whether repetition is allowed will be specified when referring to a list)

- **Size** or **cardinality**: _k-set_ (k-list) is a _set_ (list) of size _k_

- **Theorem 1.** There are \( n^k \) ways to form a k-list from elements of an n-set.  (*Order matters, with repetition*)

- **Theorem 4:** There are \( n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot (n - k + 1) \)
  \[= n! / (n - k)! = P(n,k) \] ways to form a k-list from distinct elements of an n-set.  (*Order matters, without repetition*)
Review

• How many ways are there to form a three-letter sequence (i.e., a 3-list) using the letters a, b, c, d, e, f (i.e., from elements of the 6-set \{a,b,c,d,e,f\}) with repetition allowed?
  – A: 20  B: 216  C: 120  D: 30  E: 720

• How many ways are there to form a three-letter sequence using the letters a, b, c, d, e, f without repetition allowed?
  – A: 20  B: 216  C: 120  D: 30  E: 720

• How many 20-lists without repetitions can be constructed from an 8-set?
  – A: 20!  B: 8!  C: 20! / 8!  D: 8! / 20!  E: None of the above

What if order doesn’t matter?
→ Get “subsets”, “selection” instead of “lists”, “ordering”
Theorem 2: Rule of Product

- Suppose structures are constructed by making a sequence of $k$ choices such that
  - (1) the $i^{th}$ choice can be made in $c_i$ ways, a number independent of what choices were previously made, and
  - (2) each structure arises in exactly one way in the process.

Then the number of structures is $c_1 \cdot c_2 \cdot \ldots \cdot c_k$.

- **How many different (858) area-code phone numbers are there that begin with 353-?**
  - **A:** $10^4 = 10 \cdot 10 \cdot 10 \cdot 10$
  - **B:** $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
  - **C:** $4! = 4 \cdot 3 \cdot 2 \cdot 1$
  - **D:** $10! / 6! = 10 \cdot 9 \cdot 8 \cdot 7$
  - **E:** None of the above

**How did we apply Rule of Product?**
Choose digit $x$ in 353-$x$, in any of 10 ways.
AND Choose digit $y$ in 353-$xy$, in any of 10 ways.
AND Choose digit $z$ in 353-$xyz$, in any of 10 ways.
Etc.
Theorem 2: Rule of Product

• Suppose structures are constructed by making a sequence of $k$ choices such that
  – (1) the $i^{th}$ choice can be made in $c_i$ ways, a number independent of what choices were previously made, and
  – (2) each structure arises in exactly one way in the process.
Then the number of structures is $c_1 \cdot c_2 \cdot \ldots \cdot c_k$.

• How many different (858) phone numbers are there that begin with 353- and contain no zeros?
  – **A**: $10^3 = 10 \cdot 10 \cdot 10$
  – **B**: $9! / 5! = 9 \cdot 8 \cdot 7 \cdot 6$
  – **C**: $9^4 = 9 \cdot 9 \cdot 9 \cdot 9$
  – **D**: $9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
  – **E**: None of the above

How did we apply Rule of Product?
Choose digit $x$ in 353-$x$, in any of 9 ways.
AND Choose digit $y$ in 353-$xy$, in any of 9 ways.
AND Choose digit $z$ in 353-$xyz$, in any of 9 ways.
Etc.
Theorem 2: Rule of Product

• How many different (858) phone numbers are there that begin with 353- and contain at least one zero?

\[
\{\text{#s with at least one zero}\}
= \{\text{all #'s}\} \setminus \{\text{all #s with no zeros}\}
\]

\[10^4 - 9^4 = 3439\]

*(we counted the complement)*
Theorem 2: Rule of Product

• How many subsets does set $S = \{a, b, c\}$ have?

• Subsets: $\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

• Apply Rule of Product:
  – $a \in S$ is either in or not in a given subset
  – $b \in S$ is either in or not in a given subset
  – $c \in S$ is either in or not in a given subset

A set $S$ has $2^{|S|}$ subsets
Cartesian Product

• Given sets $C_1$, $C_2$, ..., $C_k$

• Their **Cartesian product** is $C_1 \times C_2 \times \ldots \times C_k$ and has elements $(x_1, x_2, \ldots, x_k)$ where $x_i \in C_i$, $i = 1, \ldots, k$

• **Cardinalities:** $|C_1 \times C_2 \times \ldots \times C_k| = |C_1| \cdot |C_2| \cdot \ldots \cdot |C_k|$

• **Example:** $S = \{a, b, c\}; \ T = \{1, 2\}$

• $S \times T = \quad$

• $|S \times T| = \quad$
Cartesian Product and Lexicographic Order

• Given sets $C_1, C_2, \ldots, C_k$

• Their *Cartesian product* is $C_1 \times C_2 \times \ldots \times C_k$ and has elements $(x_1, x_2, \ldots, x_k)$ where $x_i \in C_i$, $i = 1, \ldots, k$

• Lexicographic order = *dictionary order*  
  (given that the elements of each set $C_i$ are ordered)

• Example: $S = \{a, b, c\}$ with $a < b < c$

• $S \times S$ in lexicographic order:
Theorem 3: Rule of Sum

• Suppose a set $T$ of structures can be partitioned into sets $T_1, \ldots, T_j$ so that each structure in $T$ appears in exactly one $T_i$. Then $|T| = |T_1| + \cdots + |T_j|$. 

• A three-person committee is formed by selecting people from departments $A$ (10 people), $B$ (12 people), $C$ (15 people) and $D$ (18 people). How many ways are there to select the committee such that no two people are from the same department?

- $A, B, D$ OR $A, B, D$ OR $A, C, D$ OR $B, C, D$
Counting: What You Need To Know

• How to count “constructively”
  – When to add == apply Rule of Sum
  – When to multiply == apply Rule of Product
• When ordering matters vs. when it doesn’t matter
  – (permutations vs. combinations)
• When to correct for over-counting
  – (e.g., when there are symmetries)
• How to break down into cases
• How to deal with a “restriction”
• When to count the complement
  – (when it’s easier to do that)
Counting Problem Examples

What if order doesn’t matter? \(\rightarrow\) subsets, selection

- How many ways to select a \(k\)-set that is a subset of a given \(n\)-set? \(=\) “combinations” \(C(n,k)\)

- Theorem 4 \(\Rightarrow\) There are \(n! / (n – k)!\) ways to select a \(k\)-list (without replacement) from a given \(n\)-set

- This over-counts by a factor of \(k!\) since each \(k\)-set can have its elements listed in \(k!\) ways (Theorem 4, again)

- \(C(n,k) = \text{number of ways to select } k \text{ out of } n \text{ elements} = n! / [(n-k)! k!] \) (Theorem 7)
Counting Problem Examples

Counting with replacement, when order does NOT matter

- **How many ways are there to order a two-scoop ice cream cone when there are five available flavors?**
  - Note: here, chocolate-vanilla is same as vanilla-chocolate

- **Cases (Rule of Sum):**

- **Staircase Method:**
Counting Problem Examples

Counting with replacement, when order does NOT matter

• How many different orders for six hot dogs are possible if there are three varieties of hot dog?

• Order Form (“Stars and Bars”) Method:
Counting Problem Examples

A Common Problem Type

• How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$, with $x_i \geq 0$?

• How do we apply the Order Form Method?

• How many solutions are there with $x_i > 0$?
Counting Problem Examples

A Common Problem Type

• How many integer solutions are there to the equation \( x_1 + x_2 + x_3 + x_4 = 12 \), with \( x_i \geq 0 \)?

• How do we apply the Order Form Method?

• How many solutions are there with \( x_1 \geq 2 \), \( x_2 \geq 2 \), \( x_3 \geq 4 \), \( x_4 \geq 0 \)?
Counting Problem Examples

Licorice

• How many ways are there to distribute 20 (identical) sticks of red licorice and 15 (identical) sticks of black licorice among five children?
Counting Problem Examples

Case analysis

• How many 4-digit campus telephone numbers have one or more repeated digits?

• Cases:
How many ways are there to pick a collection of 10 balls from a pile of red balls, blue balls, and purple balls if at least 5 red balls must be picked?

Simplification:

What if at most 5 red balls must be picked?
Problems 2: (more) Pigeonhole, Invariants

• P2.1 A dragon has 100 heads. A knight can cut off 15, 17, 20, or 5 heads with one blow of his sword. In each of these cases, 24, 2, 14, or 17 new heads, respectively, will grow back. If all heads are cut off, the dragon dies. Can the dragon ever die? Explain.

• P2.2 There are N persons in a room. Prove that among them there are two persons who have the same number of acquaintances in the room.

• P2.3 Every point in the plane is colored either red or blue. Prove that somewhere in the plane, there are three vertices of an equilateral triangle which are all the same color.

• P2.4 Inside a room of area 5, you place 9 rugs, each of area 1 and having an arbitrary shape. Explain why there must be two rugs which overlap by at least 1/9. Hint: \((1 + 2 + \ldots + 9)/9 = 5\)