Lecture 15 Notes

• Goals for this week
  – Big-O complexity (GT Section 4) (Tuesday)
  – Solving recurrences (DT Section 4) (Tuesday and today)
  – For quiz: Tuesday material
  – Get ready for graphs, graph properties, and graph algorithms! (GT Sections 1, 2, 3) (try to start today)
• Final exam review session – when?
  – Practice final exam questions – early Week 10
• Week 9 and Week 10 quizzes: going “off-line”
  – (e.g., 12-hour window to take a quiz using WeBWorK)
• “Things to Know” – end of Week 9, end of Week 10

• ** “coffee with a prof”: some of you mentioned this earlier in the quarter – if you want to follow up, we’ll need to start scheduling
  – note: my limit = six espressos per day 😊
Main Points From Last Time

• D/Q Multiplication details: \( T(n) = 4T(n/2) + O(n) \)

• Fibonacci numbers
  – fib1: exponential \( \text{solved from C.P. } a^2 - a - 1 = 0, \text{ initial conditions} \)
  – fib2: “linear” (quadratic number of bitwise multiplies)
  – fib3: “logarithmic” (idea of addition chains)

• Asymptotic resource usage: “as \( n \) grows large”
  – \( n \) = natural parameter of input (#elements, #vertices, …)

• Big-O, big-Omega, big-Theta
  – \( f \in O(g) \) if \( \exists c > 0, N \text{ s.t. } \forall n > N, f(n) \leq cg(n) \)
    to show \( f \) is \( O(g) \): exhibit \( c, N \) \( \text{e.g., } 200n^2 \leq 1 \cdot 2n^{2.5} \forall n > 10000 \)
  – \( f \in \Omega(g) \) if \( g \in O(f) \)
  – \( f \in \Theta(g) \) if \( g \in O(f) \text{ and } f \in O(g) \)
  – \( [ f \in o(g) \text{ iff } \lim_{n \to \infty} f(n)/g(n) = 0 ] \)
Using “Big-Oh” Notation – Examples

• Definition: \( f(n) \) is **monotonically growing** (non-decreasing) if \( n_1 \geq n_2 \Rightarrow f(n_1) \geq f(n_2) \)

• **Fact:** For all constants \( c > 0, a > 1, \) and for all monotonically growing \( f(n), (f(n))^c \in O(a^{f(n)}) \)

  \[ \text{e.g., } n^{1000} \text{ is } O(2^n) \]

• **Corollary** *(take \( f(n) = n \))*: \( \forall c > 0, a > 1, n^c \in O(a^n) \)
  – Any exponential in \( n \) grows faster than any polynomial in \( n \)

• **Corollary** *(take \( f(n) = \log_a n \))*: \( \forall c > 0, a > 1, (\log_a n)^c \in O(a^{\log_a n}) = O(n) \)
  – Any polynomial in \( \log n \) grows slower than \( n^{c'}, c' > 0 \)

• **Exercise:** \( f \in O(s), g \in O(r) \Rightarrow f+g \in O(s+r) \)

• **Exercise:** \( f \in O(s), g \in O(r) \Rightarrow f \cdot g \in O(s \cdot r) \)
• **Q1:** The recurrence for Mergesort runtime is \( T(n) = \) ___ ?
  
  A: \( 2T(n-1) + 1 \)  
  B: \( 4T(n/2) + O(n) \)  
  C: \( 2T(n/2) + n \)  
  D: \( 2^n - 1 \)  
  E: None of above

  \( \text{this is Towers of Hanoi!} \)

• **Q2:** The solution for Towers of Hanoi runtime is \( T(n) = \) ___ ?
  
  A: \( 2T(n-1) + 1 \)  
  B: \( n \log_2 n + n \)  
  C: \( 2T(n/2) + n \)  
  D: \( 2^n - 1 \)  
  E: None of above

  \( \text{this is Mergesort} \)

• **Q3:** If \( f \in O(g) \) and \( g \in O(h) \), then \( f \in O(h) \)
  
  A: always  
  B: sometimes, but not always  
  C: never  
  D: None of above

• **Q4:** If \( f \in O(g) \) and \( g \in O(f) \), then \( f \in \Theta(g) \)
  
  A: always  
  B: sometimes, but not always  
  C: never  
  D: None of above

\( \text{def of } f \in \Theta(g) \)

\[ f \leq c_1 g \quad \forall n > N_1 \]

\[ g \leq c_2 h \quad \forall n > N_2 \]

\[ \Rightarrow f \leq (c_1 \cdot c_2) h \quad \forall n > \max(N_1, N_2) \]
Theorem 9 (DT-48)

Theorem 9: Let $a_0, a_1, \ldots, a_n$ be a sequence of numbers. Suppose there are constants $b$ and $c$ such that

$$a_n = ba_{n-1} + ca_{n-2} \text{ for } n \geq 2.$$ 

Let $r_1$ and $r_2$ be the roots of the characteristic equation $r^2 - br - c = 0$. Also:

- characteristic polynomial $\alpha^2 = br + c$
- characteristic equation $\alpha^2 - b\alpha - c = 0$

If there are two distinct real roots, $r_1, r_2$:

$$a_n = \alpha r_1^n + \beta r_2^n \text{ for } n \geq 0$$

where: $a_0 = \alpha + \beta$ and $a_1 = r_1\alpha + r_2\beta$

If there is one repeated real root, $r$:

$$a_n = \alpha r^n + \beta nr^n \text{ for } n \geq 0$$

where: $a_0 = \alpha$ and $a_1 = r\alpha + r\beta$

"initial conditions" $a_0, a_1$

2 equations in 2 unknowns $\alpha, \beta$
Example 1

- Find the exact solution to the recurrence equation:
  \[ a_n = a_{n-1} + 2a_{n-2}, \text{ where } a_0 = 1 \text{ and } a_1 = 8 \]

\[ r^n = r^{n-1} + 2r^{n-2} \]
\[ r^2 = r + 2 \]
\[ r^2 - r - 2 = 0 \]

C.P. = \[ r^2 - r - 2 = 0 \]

I.C.'s

\[ a_n = b a_{n-1} + c a_{n-2} \]
\[ b = 1, \ c = 2 \]
Example 1

• Find the exact solution to the recurrence equation:
  \[ a_n = a_{n-1} + 2a_{n-2} \], where \( a_0 = 1 \) and \( a_1 = 8 \)

• Recall: the characteristic equation for \( a_n = ba_{n-1} + ca_{n-2} \) is
  \[ r^2 - br - c = 0 \]

• Our characteristic equation is?
Example 1

• Find the exact solution to the recurrence equation:
  \[ a_n = a_{n-1} + 2a_{n-2}, \text{ where } a_0 = 1 \text{ and } a_1 = 8 \]

• Recall: the characteristic equation for \( a_n = ba_{n-1} + ca_{n-2} \) is
  \[ r^2 - br - c = 0 \]

• Our characteristic equation is?
  \[ r^2 - r - 2 = 0 \]

\[ (r-2)(r+1) = 0 \]

roots \( r=2, -1 \)
Example 1

- Find the exact solution to the recurrence equation: \( a_n = a_{n-1} + 2a_{n-2} \), where \( a_0 = 1 \) and \( a_1 = 8 \)

- Recall: the characteristic equation for \( a_n = ba_{n-1} + ca_{n-2} \) is \( r^2 - br - c = 0 \)

- Our characteristic equation is? \( r^2 - r - 2 = 0 \)

- Solving for the roots:
  \( r^2 - r - 2 = 0 \)
  \( (r - 2)(r + 1) = 0 \)
  \( r_1 = 2, \ r_2 = -1 \)

- \( a_n = \alpha(2)^n + \beta(-1)^n \)

Need to match I.C.'s with appropriate \( \alpha, \beta \).
Example 1 Continued

• Find the exact solution to the recurrence equation:
  \[ a_n = a_{n-1} + 2a_{n-2}, \text{ where } a_0 = 1 \text{ and } a_1 = 8 \quad \text{(I.C.'s)} \]

• Theorem 9 tells us that if there are two real roots:
  \[ a_n = \alpha r_1^n + \beta r_2^n \text{ for } n \geq 0 \]
  where \( a_0 = \alpha + \beta, a_1 = r_1\alpha + r_2\beta \)

Here: \( r_1 = 2, r_2 = -1 \) \( \Rightarrow \) \[ a_n = \alpha 2^n + \beta (-1)^n \]
Example 1 Continued

• Find the exact solution to the recurrence equation:
  \[ a_n = a_{n-1} + 2a_{n-2}, \text{ where } a_0 = 1 \text{ and } a_1 = 8 \]

• Theorem 9 tells us that if there are two real roots:
  \[ a_n = \alpha r_1^n + \beta r_2^n \text{ for } n \geq 0 \]
  where \( a_0 = \alpha + \beta \), \( a_1 = r_1\alpha + r_2\beta \)
  Since here \( r_1 = 2 \), \( r_2 = -1 \) \( \Rightarrow a_n = \alpha 2^n + \beta n(-1)^n \)

• Plugging into equations for \( a_0 \) and \( a_1 \), solving for \( \alpha \) and \( \beta \):
  
  \[ 1 = \alpha + \beta \]
  
  \[ 8 = 2\alpha - \beta \]

  Add: \( 9 = 3\alpha \) \( \Rightarrow \alpha = 3 \) \( \Rightarrow \beta = -2 \)

• Putting it all together:
  \[ a_n = 3(2^n) - 2(-1)^n \]
Proof of Correctness by Induction on \( n \)

- Trying to prove: \( a_n = 3(2^n) - 2(-1)^n \)
- Given: \( a_n = a_{n-1} + 2a_{n-2}, \; a_0 = 1, \; a_1 = 8 \)

- **Base case** (does our equation work for \( a_0 \) and \( a_1 \)?):
  - \( a_0 = 3(2^0) - 2(-1)^0 = 1 \) ✔ ✔
  - \( a_1 = 3(2^1) - 2(-1)^1 = 8 \) ✔ ✔

- **Inductive step** (does it work for \( a_n \), when \( a_n = a_{n-1} + 2a_{n-2} \)?):
  - \( a_n = a_{n-1} + 2a_{n-2} \)
  - \( = 3(2^{n-1}) - 2(-1)^{n-1} + 2(3(2^{n-2}) - 2(-1)^{n-2}) \)
  - \( = 6(2^{n-2}) - 4((-1)^{n-2}) + 3(2^{n-1}) - 2(-1)^{n-1} \)
  - \( = 3(2^{n-1}) + 4((-1)^{n-1}) + 3(2^{n-1}) - 2(-1)^{n-1} \)
  - \( = 6(2^{n-1}) + 2((-1)^{n-1}) \)
  - \( = 3(2^n) - 2((-1)^n) \) ✔

Prove this by induction.

I have the correct solution to induction + I.C.'s.
Example 2: Gambler’s Ruin

- A gambler repeatedly bets a flipped coin will come up heads.
  - If the coin is heads, the gambler wins $1.
  - If the coin is tails, the gambler loses $1.
  - If the gambler ever reaches $M he/she will stop.

- Let $P_k =$ probability gambler loses all $k$ he/she has (= “ruin”)

$$P_k = P(H)*P(\text{ruin}|H) + P(T)*P(\text{ruin}|T)$$

$$P_k = \frac{1}{2} \cdot P(\text{ruin}|H) + \frac{1}{2} \cdot P(\text{ruin}|T) \quad (\text{since we’re flipping a coin})$$

$$P_k = \frac{1}{2} \cdot P(\text{ruin}|\text{win $1$}) + \frac{1}{2} \cdot P(\text{ruin}|\text{lose $1$})$$

$$P_k = \frac{1}{2} \cdot P_{k+1} + \frac{1}{2} \cdot P_{k-1}$$

$$-\frac{1}{2}P_{k+1} = -P_k + \frac{1}{2}P_{k-1} \Rightarrow P_{k+1} = 2P_k - P_{k-1} \Rightarrow P_k = 2P_{k-1} - P_{k-2}$$
Example 2: Gambler’s Ruin continued

- We just learned: \( P_k = 2P_{k-1} - P_{k-2} \)

- Recall: the characteristic equation for \( a_n = ba_{n-1} + ca_{n-2} \) is \( r^2 - br - c = 0 \)

- Here our characteristic equation is:
  \[ r^2 - 2r + 1 = 0 \]
  \( (r-1)(r-1) = 0 \)
  So we have the repeated root, \( r = 1 \)

- We know \( P_0 = 1 \) (if we start with $0 we’re already ruined)
- We know \( P_M = 0 \) (if we start with $M we quit playing the game)
**Example 2: Gambler’s Ruin continued**

- We just learned: \( P_k = 2P_{k-1} - P_{k-2} \) and \( P_0 = 1, P_M = 0 \)

- Theorem 9: if there is one repeated real root, \( r \):
  \[
  a_n = \alpha r^n + \beta nr^n \quad \text{for} \quad n \geq 0
  \]
  where: \( a_0 = \alpha \)
  \[
  a_1 = r\alpha + r\beta
  \]
  remember: we have \( r = 1 \)

- \( 0 = P_M \)
  \[
  P_M = \alpha r^M + \beta M r^M = P_0 + \beta M
  \]
  \[
  \Rightarrow 0 = 1 + \beta M \quad \Rightarrow -1 = \beta M
  \]
  \[
  \Rightarrow \beta = -1/M
  \]
  \[
  \Rightarrow P_n = 1 - n/M
  \]
Example 2: Gambler’s Ruin continued

- \( P_n = 1 - \frac{n}{M}, \) so our probability of ruin, \( P_k = 1 - \frac{k}{M} \)

- If we have $10 and won’t stop playing unless we have $100, what is the probability that we will lose our $10?
  \[
  k = 10 \\
  M = 100 \\
  P_{10} = 1 - \frac{10}{100} = .90, \text{ or } 90\%
  \]

- (Life advice! 😊)

- If we have $10 but only want to win $12, what is the probability that we will lose our initial $10?
  \[
  P_{10} = 1 - \frac{10}{12} = 1/6 = 0.1667, \text{ now only } 16.67\%)
  \]


Linear Recurrence Relations

• An n\textsuperscript{th} order linear homogeneous recurrence relation with constant coefficients (lhcc) is a recurrence relation of the form:
  
  \[ a_k = A_1 a_{k-1} + A_2 a_{k-2} + \ldots + A_n a_{k-n} \text{ with } A_k \in \mathbb{R}, A_n \neq 0 \]

• A linear non-homogeneous recurrence relation with constant coefficients (lnhcc) is a recurrence relation of the form:

  \[ a_k = A_1 a_{k-1} + A_2 a_{k-2} + \ldots + A_n a_{k-n} + F(k) \]

  where \( A_1, \ldots, A_n \) are real numbers

  and \( F(k) \) is a function depending only on \( k \)

  \[ a_n = (\text{soln to homogeneous piece}) + (\text{particular soln}) \]
Recipe

• When solving LHCCs and LNHCCs in this class:

\[ a_k = A_1 a_{k-1} + A_2 a_{k-2} + \ldots + A_n a_{k-n} \]
\[ a_k = A_1 a_{k-1} + A_2 a_{k-2} + \ldots + A_n a_{k-n} + F(k) \]

can use characteristic equation:
\[ r^n - A_1 r^{n-1} - A_2 r^{n-2} - \ldots - A_n = 0 \]

and \[ a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_2 r^n + \beta F(k) \]
Recipe

• When solving LHCCs and LNHCCs in this class:
  
  \[ a_k = A_1 a_{k-1} + A_2 a_{k-2} + \ldots + A_n a_{k-n} \]
  
  \[ a_k = A_1 a_{k-1} + A_2 a_{k-2} + \ldots + A_n a_{k-n} + F(k) \]

  can use characteristic equation: \( r^n - A_1 r^{n-1} - A_2 r^{n-2} - \ldots - A_n = 0 \)

  and \( a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_r r^n + (\beta F(k)) \)

• When solving LNHCCs:

  • If \( F(n) \) is a polynomial in \( n \), try polynomial of same degree as the particular solution

  (see example...)

  • If \( F(n) \) is exponential with base different from roots of characteristic equation, try constant multiple of \( F(n) \) as particular solution

  (see example)

  • First solve for particular solution constants, then use the initial conditions to find homogeneous solution constants
LHNCC example

• Find an explicit solution to the recurrence equation:
  \[ a_n = 5a_{n-1} - 6a_{n-2} + 4^n \quad a_0 = 0, \quad a_1 = 1 \]

• Characteristic equation form:
  \[ r^n - A_1 r^{n-1} - A_2 r^{n-2} - \ldots - A_n = 0 \quad \text{with} \quad a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_2 r_n^n \quad (+ \beta F(k) ) \]

• Here we have:
  \[ r^2 - 5r + 6 = (r - 2)(r - 3), \text{ so } r_1 = 2 \text{ and } r_2 = 3 \]
  \[ a_n = \alpha_1 2^n + \alpha_2 3^n + \beta 4^n \]

• What is the constant, \( \beta \), in the particular solution?
LNHCC example continued

- Given $a_n = 5a_{n-1} - 6a_{n-2} + 4^n$, $a_0 = 0$, $a_1 = 1$
- Now we know: $a_n = \alpha_1 2^n + \alpha_2 3^n + \beta 4^n$

- What is the constant, $\beta$, in the particular solution?
  - Suppose $a_n = \beta 4^n$
  - Then $\beta 4^n = 5\beta 4^{n-1} - 6\beta 4^{n-2} + 4^n$
  - So $\beta = (5/4) \beta - (6/16) \beta + 1$

  \[ \Rightarrow \beta = 8 \]

- Thus, the form of the general solution is:
  \[ a_n = \alpha_1 2^n + \alpha_2 3^n + 8(4^n) \]
LNHCC example continued

• Given $a_n = 5a_{n-1} - 6a_{n-2} + 4^n$, $a_0 = 0$, $a_1 = 1$

• Now we know: $a_n = \alpha_1 2^n + \alpha_2 3^n + 8(4^n)$

• Plugging in values:
  
  $0 = a_0 = \alpha_1 2^0 + \alpha_2 3^0 + 8(4^0) = \alpha_1 + \alpha_2 + 8$

  $1 = a_1 = 2\alpha_1 + 3\alpha_2 + 32$

• Subtracting twice the top from the bottom gives:
  
  $1 = \alpha_2 + 16$, so $\alpha_2 = -15$

  and $0 = \alpha_1 + \alpha_2 + 8$, so $\alpha_1 = 7$

General solution is: $a_n = 7(2^n) - 15(3^n) + 8(4^n)$
Proof of Solution By Induction

• To prove: General solution for \( a_n = 5a_{n-1} – 6a_{n-2} + 4^n \), \( a_0 = 0 \), \( a_1 = 1 \) is

\[
a_n = 7(2^n) – 15(3^n) + 8(4^n)
\]

• Base case:

\( (n = 0) : 7 – 15 + 8 = 0 = a_0 \) ✔

\( (n = 1) : 7 \cdot 2 – 15 \cdot 3 + 8 \cdot 4 = 1 = a_0 \) ✔

• Induction step:

\[
5a_{n-1} – 6a_{n-2} + 4^n
= 5(7(2^{n-1}) – 15(3^{n-1}) + 8(4^{n-1})) – 6(7(2^{n-2}) – 15(3^{n-2}) + 8(4^{n-2})) + 4^n
= 2^n(5 \cdot 7/2 – 6 \cdot 7/4) + 3^n(-15 \cdot 5/3 + 6 \cdot 15/9) + 4^n(5 \cdot 8/4 – 6 \cdot 8/16 + 1)
= 2^n(7) – 3^n(15) + 4^n(8)
= a_n
\]
„Master Theorem for Recursions“
Theorem 8, GT-47

Recurrence \( T(n) \leq a \cdot T(n/b) + O(n^d) \)

1) \( T(n) = O(n^d) \) if \( a < b^d \)
2) \( T(n) = O(n^d \log n) \) if \( a = b^d \)
3) \( T(n) = O(n^{\log_b a}) \) if \( a > b^d \)

Work done at \( k \)th level is \( a^k \cdot O(n/b^k)^d = O(n^d) \cdot (a/b^d)^k \)

Type (3): long multiplication, matrix multiplication

Type (2): mergesort
Master Theorem Examples

• Mergesort $T(n) = 2T(n/2) + \Theta(n)$

$$T(n) = \Theta(n \log n)$$

• Matrix Multiply $T(n) = 8T(n/2) + \Theta(n^2)$

$$T(n) = \Theta(n^{\log_2 8}) \approx \Theta(n^3)$$
Proof of “Master Theorem”

This is CSE 101 material: don’t worry for now, but you’ll need this in a year.

Recurrence: \( T(n) \leq a \cdot T(n/b) + O(n^d) \)

1) \( T(n) = O(n^d) \) if \( a < b^d \)
2) \( T(n) = O(n^d \log n) \) if \( a = b^d \)
3) \( T(n) = O(n^{\log_b a}) \) if \( a > b^d \)

- Total work done at \( k^{th} \) level is \( a^k \times O(n/b^k)^d = O(n^d) \times (a / b^d)^k \)
- As \( k \) goes from 0 (root) to \( \log_b n \) (leaves), have geometric series with ratio \( a / b^d \)

- Do you remember how to sum a geometric series?
  - \( a / b^d < 1 \rightarrow \) sum is \( O(n^d) \)
  - \( a / b^d > 1 \rightarrow \) sum is given by last term, \( O(n^{\log_b a}) \)
  - \( a / b^d = 1 \rightarrow \) sum is given by \( O(\log n) \) terms equal to \( O(n^d) \)
# Master Theorem, Extended a Bit ...

<table>
<thead>
<tr>
<th>$a$, $b$</th>
<th>$f(n)$ ($=n^d$)</th>
<th>$T(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 1$</td>
<td>$c$</td>
<td>$\log_d n$</td>
</tr>
<tr>
<td>$a = b$</td>
<td>$c$</td>
<td>$A_1 n + A_2$</td>
</tr>
<tr>
<td>$a &lt; b$</td>
<td>$cn$</td>
<td>$A_1 n$</td>
</tr>
<tr>
<td>$a = b$</td>
<td>$cn$</td>
<td>$O(n \log_{b} n)$</td>
</tr>
<tr>
<td>$a &gt; b$</td>
<td>$cn$</td>
<td>$O(n \log_{b} a)$</td>
</tr>
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(From Tucker’s *Applied Combinatorics* text)
The MaxMin Problem

• Given a set S of n numbers, use divide-and-conquer to find the maximum element and the minimum element in S

  Split S into two subsets of size n/2 each
  Recursively find the max and min of each subset
  Compare the two max’s, compare the two min’s

• Recurrence: \( T(n) = 2T(n/2) + 2 \)

• Which case is this in the previous slide? \( T(n) = 3n/2 - 2 \)
Graphs
(GT Sections 1, 2)

- A graph is a model of relationships between pairs of objects.
  - Objects: nodes / vertices
  - Relationships: edges / links / arrows

\[
G = (V, E)
\]

\[P_2(V) = \text{set of 2-subset of } V\]

<table>
<thead>
<tr>
<th>Type</th>
<th>Description</th>
<th>GT Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple graph</td>
<td>((V, E)) - Undirected - ((E \subseteq P_2(V))) - No self loops - No edge labels or weights</td>
<td>GT-2</td>
</tr>
<tr>
<td>Graph</td>
<td>((V, E, \phi)) - Undirected - ((E \subseteq P_2(V))) - No self loops - Has edge labels</td>
<td>GT-3</td>
</tr>
<tr>
<td>Simple graph with loops</td>
<td>((V, E, \phi)) - Undirected - ((E \subseteq P_2(V))) - May have self loops - Has edge labels</td>
<td>GT-4</td>
</tr>
<tr>
<td>Digraph</td>
<td>((V, E, \phi)) - Directed - ((E \subseteq V \times V)) - May have self loops - Has edge labels</td>
<td>GT-15</td>
</tr>
</tbody>
</table>
Graph examples

- Facebook
  - Nodes: people
  - Edges: friendships
  - Simple graph
    - Dynamic evolves over time

- Printed circuit board
  - Nodes: transistors, resistors, etc.
  - Edges: wires
  - Simple graph

- Hypergraph
  \[ H = (V, E) \]
  - \( e \in E \) is a subset of \( V \), but \( |e| \geq 2 \) is allowed

in circuits, a
sign "net" from
pin A to pins B, C, D
is not really a graph
edge... (it’s a directed hyperedge
in a hypergraph...
Graph examples

- Organizational chart
  - **Nodes:** roles
  - **Edges:** management
  - Directed graph

- Internet routing
  - **Nodes:** routers
  - **Edges:** links
  - Directed graph

WWW.Caida.org is right here on campus!!!
Representing graphs

- **Adjacency list**: for each vertex, list neighbors
- **Adjacency matrix**: $A_{ij} = 1$ iff vertices $i$ and $j$ are adjacent

**List:**

- $1 : \{2, 4\}$
- $2 : \{1, 3, 4\}$
- $3 : \{2\}$
- $4 : \{1, 2\}$

**Matrix:**

$$
\begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
$$

- $A_{14} = A_{41} = 1$
- $A_{23} = A_{32} = 1$

(symmetric adjacency matrix for an undirected graph...
Graph Properties

• What is the maximum number of edges in a simple graph with \( n \) vertices?
  
  A. \( n \)
  B. \( n^2 \)
  C. \( 2^n \)
  D. \( \binom{n}{2} \)
  E. None of the above.

A complete graph on \( n \) vertices is one with \( \binom{n}{2} = \frac{n(n-1)}{2} \) edges.

Will start here next time!
Graph Properties

• If we start with n vertices and, for each pair of distinct vertices decide randomly (by flipping a fair coin) whether to place an edge between them, what is the expected number of edges in the graph?

  A. n/2
  B. n^2 – 1
  C. 2^{n-1}
  D. n(n−1)/4
  E. None of the above.

• Recall from homework: This is a random graph – specifically, a graph in G(n, 0.5).
Degree of a Vertex, Degree Sequence

- The **degree** of vertex $v$ in graph $G$ is the number of edges incident to $v$.
- The **degree sequence** of a graph $G$ is the list of all degrees of vertices in $G$, sorted in non-decreasing order.

What is the degree sequence of this graph?

A. $1,2,3,4$
B. $1,2,2,3$
C. $2,3,1,2$
D. $1,2,3$
E. None of the above
Properties of Degree Sequences

• Theorem: If $G = (V,E)$ is a simple graph

$$\sum_{v \in V} d(v) = 2 \times |E|$$

• Every edge has two “endpoints”
• Sometimes called “handshake lemma”

• Consequences
  – The sum of vertex degrees is even
  – The number of odd-degree vertices is even
Isomorphic graphs

- Informally, two graphs are isomorphic if you can transform the picture of one into the other by “sliding vertices around and bending, stretching, and contracting edges as need” and possibly relabeling the vertices.

- Which pair of graphs is isomorphic?
  
  A. I and III  
  B. II and IV  
  C. III and IV  
  D. I and II  
  E. None of the above.
Properties of isomorphic graphs

• If two graphs are isomorphic, then they have the same
  – numbers of vertices and number of edges
  – degree sequences
  – numbers of connected components
Problems 15

• P12.3 ( typo corrected) Let \( n = 2^k \). Prove that one can select \( n \) integers from any \( (2n - 1) \) integers such that their sum is divisible by \( n \).

• P15.1 In a certain process, at every step a pile of stones is divided into two smaller piles, and the product of the sizes of the two piles is added into a running sum. Initially, there are \( n \) stones in a single pile, and the process terminates when no more piles can be divided into smaller stones. Prove that the final value of the running sum will always be a certain fixed value, irrespective of the way in which piles made during the process.

• P15.2 A finite set \( S \) of points in the plane has the property that any line through two of the points passes through a third point. Prove that all points of \( S \) lie on a single line.
Problems 15

• P15.3 (you must give solutions to both (a) and (b) for credit. (a) Prove, using mathematical induction, that all numbers of the form 1007, 10017, 100117, 1001117, … are divisible by 53. (b) Prove, using mathematical induction, that all numbers of the form 12008, 120308, 1203308, … are divisible by 19.

• P15.4 There are 20 researchers at a famous lab. They have a meeting every week. There is a rule that if any researcher finds that he has made mistakes in his own papers, then he must resign. Over the course of many years, each researcher finds mistakes in each of his colleagues’ works, but no researcher resigns. One day during a weekly meeting, a summer intern from UCSD announces to everyone that “at least one person has made a mistake that was discovered by someone else”. Twenty meetings later, all of the researchers resign. Explain why.
Problems 15

- P15.5 The numbers 1 through \( n^2 \) are placed in the squares of an \( n \times n \) checkerboard. Prove that there exist two adjacent squares (horizontally, diagonally, vertically) whose values differ by at least \( n + 1 \).
Complexity of DQ for Long Multiplication

• This slide: illustrate “unrolling” of recurrence or, “substitution”
  – First line of attack if no convenient theorem available…

• Multiply two 2s-digit numbers, \([wx] \cdot [yz]\)
  – 4 \(n/2\)-digit multiplications: \(xz, wz, xy, wy\)
  – Digit-shifting: multiplication by \(10^s, 10^{2s}\)
  – 3 additions

\[
T(n) = 4T(n/2) + \theta(n)
\]

• \(T(n) \leq 4T(n/2) + cn\)
  \[\leq 4 [ 4T(n/4) + cn/2] + cn\]
  \[= 16 T(n/4) + (1 + 2)cn\]
  \[\leq 16 [ 4T(n/8) + cn/4] + (1 + 2)cn\]
  \[= 64 T(n/8) + (1 + 2 + 4)cn\]
  ...

\[\leq 4^k T(n/2^k) + (1 + 2 + 4 + \ldots + 2^{k-1})cn\]

For \(k = \log_2 n\):

\[T(n) \leq n^2 T(1) + cn^2 = O(n^2)\]

\(O(n^2)\) makes sense
Proof of “Master Theorem”

This is CSE 101 material: don’t worry for now, but you’ll need this in a year

Recurrence \( T(n) \leq a \cdot T(n/b) + O(n^d) \)

1) \( T(n) = O(n^d) \) if \( a < b^d \)

2) \( T(n) = O(n^d \log n) \) if \( a = b^d \)

3) \( T(n) = O(n^{\log_b a}) \) if \( a > b^d \)

- Assume \( n \) is a power of \( b \) → can ignore rounding in \( \lceil n/b \rceil \)
- Subproblem size decreases by factor of \( b \) with each level of recursion → reaches base case after \( \log_b n \) levels
- Branching factor \( a \) → \( k^{th} \) level of tree has \( a^k \) subproblems each of size \( n/b^k \)
- Total work done at \( k^{th} \) level is \( a^k \times O(n/b^k)^d = O(n^d) \times (a / b^d)^k \)
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- Total work done at \( k^{th} \) level is \( a^k \cdot O(n/b^k)^d = O(n^d) \cdot (a / b^d)^k \)
- As \( k \) goes from 0 (root) to \( \log_b n \) (leaves), have geometric series with ratio \( a / b^d \)
  - do you remember how to sum a geometric series?
    - \( a / b^d < 1 \) \( \rightarrow \) sum is \( O(n^d) \)
    - \( a / b^d > 1 \) \( \rightarrow \) sum is given by last term, \( O(n^{\log_b a}) \)
    - \( a / b^d = 1 \) \( \rightarrow \) sum is given by \( O(\log n) \) terms equal to \( O(n^d) \)