Lecture 14 Notes

• Goals for this week
  – Big-O complexity (GT Section 4)
  – Solving recurrences (DT Section 4)
  – For quiz: Tuesday material

• Feedback survey (anonymous google doc)
  – Thanks for your responses!
  – 131 responses … your lowest quiz score will be dropped

• “Midterm consolidation” (73 participated)
  – Mean after capping at 100: 80.9
  – Standard deviation: 18.15
  – If you have low scores on both original MT and this MC, please stop by OHs or otherwise arrange to see me
Main Points From Last Time

• **D/Q Mergesort**
  – Recurrence: $T(n) = 2T(n/2) + n$ = time to sort n elements
  – Solution: $T(n) = n \cdot \log_2 n + n$ proved by mathematical induction
  – Proved by mathematical induction
    • Also intuitive from recursion tree: (log n levels) * (n time/level)
  – Other D/Q: matrix multiplication: $T(n) = 8T(n/2) + n^2$
  – Other D/Q: long multiplication: $T(n) = 4T(n/2) + 3n$

• **Tower of Hanoi**
  – Recursion: $T(n) = 2T(n-1) + 1$ = time to move n disks
    • Two inequalities $\leq$, $\geq$ gave us the equality
  – Solution: $T(n) = 2^n - 1$ proved by mathematical induction
  – Other recursion: determinant: $T(n) = n \cdot T(n-1)$

• **Basketball**: #ways to score n pts
  $$= (n+1)^{st} \text{ Fibonacci number}$$
Long Multiplication in Pete’s Discussion

- Goal: multiply $1980 \times 2315$ using a D/Q approach
- Let $a = 1980$, $b = 2315$
- Let $a_L = 19$, $a_R = 80$, $b_L = 23$, $b_R = 15$ (split into left, right halves)

\[
\begin{array}{c}
  a_L \ a_R \\
  \times \ b_L \ b_R \\
  \hline
  a_L b_R \ a_R b_R \\
  + \ a_L b_L \ a_R b_L \quad (4 \text{ multiplies, each with half the digits}) \\
  \hline
  = a_L b_L (a_L b_R + a_R b_L) \ a_R b_R
\end{array}
\]
1980 × 2315 = ?

\[
\begin{array}{c}
| 19 | 80 \\
| 23 | 15 \\
\end{array}
\]

\[
(19)(15) \quad (80)(15)
\]
\[
(19)(23) \quad (23)(80)
\]

\[
\begin{array}{c}
437 \\
2125 \\
1200
\end{array}
\]

\[
4370000 + 212500 + 1200
\]

\[
= 4583700 \quad (= 1980 \times 2315)
\]
Last Week: Basketball Before You Were Born

- No 3-point field goal
- Hypothetical game score: UCSD 75, UCLA 64
- Assuming no 3-pointers, in how many ways could UCSD have accumulated 75 points?

**Notation:**
- $S(n) \equiv \# \text{ ways to score } n \text{ points}$

**Small Cases:**
- $S(0) = 1$
- $S(1) = 1$
- $S(2) = 2$ (2 or 1-1)
- $S(3) = 3$ (2-1 or 1-2 or 1-1-1)

Is this familiar?

#ways to reach $n$

$= \#\text{ways to reach } n-2 \text{ (plus a field goal)}$

$+ \#\text{ways to reach } n-1 \text{ (plus a free throw)}$

$\Rightarrow$ Fibonacci numbers!
Recurrence Relation

- $S(n) = S(n-1) + S(n-2)$

- What is $S(75)$?
  - $S(0) = 1$, $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 5$, $S(5) = 8$, …

- Fibonacci numbers $F(n)$: 1, 1, 2, 3, 5, 8, …
  - $S(75) = F(76) = 76^{th}$ Fibonacci number
Choosing Between Solutions (Algorithms)

• Criteria:
  – Correctness
  – Time resources
  – Hardware resources
  – Simplicity, clarity (practical issues)

• Will need:
  – Size (“n”, number of bits, …), Complexity measures
  – Notion of “basic” (“unit-cost”) machine operation
Fibonacci Numbers

- \text{fib1}(n) \quad \text{if } n < 2 \text{ then return } n
  
  \text{else return } \text{fib1}(n-1) + \text{fib1}(n-2)

- Analysis: \( T(n) = 1 \) if \( n < 2 \); \( T(n) = T(n-1) + T(n-2) \) otherwise
  
  \( T(n) = F(n) \)  

next two slides: this is \( \sim (1.64)^n \)
Solving the Fibonacci Recurrence

DT Theorem 9, page DT-48

• **Notation:** $F(n) = F(n-1) + F(n-2)$

• **Guess:** try $F(n) = a^n$ for some $a$

\[
a^n = a^{n-1} + a^{n-2} \quad \Rightarrow \quad a^2 = a + 1
\]
\[
\Rightarrow \quad a^2 - a - 1 = 0
\]

Roots of quadratic: $a_1 = (1 + \sqrt{5})/2$; $a_2 = (1 - \sqrt{5})/2$

*What’s missing?*
Solving the Fibonacci Recurrence

DT Theorem 9, page DT-48

• **Guess:** try \( F(n) = a^n \) for some \( a \)

\[
a^n = a^{n-1} + a^{n-2} \quad \Rightarrow \quad a^2 = a + 1
\]

\[
\Rightarrow \quad a^2 - a - 1 = 0
\]

Roots of quadratic: \( a_1 = (1 + \sqrt{5})/2; \ a_2 = (1 - \sqrt{5})/2 \)

• **Use all of the information**

  We know that \( F(1) = 1; \ F(2) = 1 \) (initial conditions)

• **Theorem 9: Homogeneous linear recurrence:**
  any linear combination of \( (a_1)^n, (a_2)^n \) is a solution

  – Set up two equations in two unknowns:

  \[
  c_1 (a_1)^1 + c_2 (a_2)^1 = F(1) = 1 \ ; \ c_1 (a_1)^2 + c_2 (a_2)^2 = F(2) = 1
  \]

  \[
  \Rightarrow c_1 = 1 / \sqrt{5} , \ c_2 = -1 / \sqrt{5}
  \]

  \[
  \Rightarrow F(n) = c_1 (a_1)^n + c_2 (a_2)^n
  \]
Fibonacci Numbers

- fib2(n)  
  \[ f[1] = 1; \ f[2] = 2; \]
  for \( j = 3 \) to \( n \) do
    \[ f[j] = f[j - 1] + f[j - 2] \]

- Analysis:  \( T(n) = n \)
  - Saving your work ("caching") can be useful!

- .... But… note that \#bits in F(n) is \( \sim 0.694n \)
  - \#bits is linear in \( n \) because F(n) is exponential in \( n \)
  - So, if we count bit operations, need quadratic number of bitwise additions to get F(n)
    
    *Always need to understand “what is a unit-cost operation”!* 

- ***Fibonacci: Can we do “better”?***
Not Obvious, But Here Is A Shortcut…

• \text{fib3}(n)
  – Consider 2x2 matrix \( M \): \( m_{11} = 0, m_{12} = 1, m_{21} = 1, m_{22} = 1 \)

  – Observe: \( [F(k) \ F(k+1)]^T = M \times [F(k-1) \ F(k)]^T \)
    \[
    [F(n+1) \ F(n+2)]^T = M^n \times [F(1) \ F(2)]^T = M^n \times [1 \ 1]^T
    \]

  – \text{How does this help?}

  – \text{Hint: } 76_{10} = 1001100_2

• \( M^{76} = M^{64} \times M^8 \times M^4 \)

  \( \Rightarrow \) \text{fib3 uses “addition chains”}
Quantifying “Better”, “Worse”

- Resources used in computation often depend on a natural parameter, $n$, of the input
  - search/sort list # items $x > y$
  - matrix mult largest dim $x \cdot y ; x + y$
  - traverse tree # nodes follow ptr

- Asymptotic Notation “as $n$ grows large”
  - $f \in O(g)$ if $\exists c > 0, N$ s.t. $\forall n > N, f(n) \leq cg(n)$
  - e.g., $200n^2 \in O(2n^{2.5})$
  - e.g., $2n + 20 \in O(n^2)$
  - “$f$ grows no faster than $g$”
  - $f \in \Omega(g)$ if $g \in O(f)$
  - $f \in \Theta(g)$ if $g \in O(f)$ and $f \in O(g)$
  - $[ f \in o(g) \text{ iff } \lim_{n \to \infty} f(n)/g(n) = 0 ]$
Theorem 9 (DT-48)

Theorem 9: Let $a_0, a_1, \ldots, a_n$ be a sequence of numbers. Suppose there are constants $b$ and $c$ such that
\[ a_n = ba_{n-1} + ca_{n-2} \text{ for } n \geq 2. \]

Let $r_1$ and $r_2$ be the roots of the characteristic equation $r^2 - br - c = 0$. Also: characteristic polynomial

If there are two distinct real roots, $r_1, r_2$:
\[ a_n = \alpha r_1^n + \beta r_2^n \text{ for } n \geq 0 \]
where: $a_0 = \alpha + \beta$ and $a_1 = r_1\alpha + r_2\beta$

If there is one repeated real root, $r$:
\[ a_n = \alpha r^n + \beta nr^n \text{ for } n \geq 0 \]
where: $a_0 = \alpha$ and $a_1 = r\alpha + r\beta$
Example 1

- Find the exact solution to the recurrence equation:
  \[ a_n = a_{n-1} + 2a_{n-2}, \]  where \( a_0 = 1 \) and \( a_1 = 8 \)
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• Recall: the characteristic equation for \( a_n = ba_{n-1} + ca_{n-2} \) is
  \[ r^2 - br - c = 0 \]

• Our characteristic equation is?
Example 1

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  \[ r^2 - r - 2 = 0 \]
Example 1

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• Recall: the characteristic equation for \( a_n = ba_{n-1} + ca_{n-2} \) is
  \[ r^2 - br - c = 0 \]

• Our characteristic equation is?
  \[ r^2 - r - 2 = 0 \]

• Solving for the roots:
  \[ r^2 - r - 2 = 0 \]
  \[ (r - 2)(r + 1) = 0 \]
  \[ r_1 = 2, \quad r_2 = -1 \]
Example 1 Continued

- Find the exact solution to the recurrence equation:
  \[ a_n = a_{n-1} + 2a_{n-2}, \text{ where } a_0 = 1 \text{ and } a_1 = 8 \]

- Theorem 9 tells us that if there are two real roots:
  \[ a_n = \alpha r_1^n + \beta r_2^n \text{ for } n \geq 0 \]
  where \( a_0 = \alpha + \beta \), \( a_1 = r_1 \alpha + r_2 \beta \)
  Since here \( r_1 = 2, r_2 = -1 \): \( a_n = \alpha 2^n + \beta n(-1)^n \)
Example 1 Continued

• Find the exact solution to the recurrence equation:
  \[ a_n = a_{n-1} + 2a_{n-2}, \text{ where } a_0 = 1 \text{ and } a_1 = 8 \]

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  Since here \( r_1 = 2, \ r_2 = -1 \): \( a_n = \alpha 2^n + \beta n(-1)^n \)

• Plugging into equations for \( a_0 \) and \( a_1 \), solving for \( \alpha \) and \( \beta \):
  \[ 1 = \alpha + \beta \]
  \[ 8 = 2\alpha - \beta \]
  Add: \( 9 = 3\alpha \), so \( \alpha = 3, \ \beta = -2 \)
Example 1 Continued

• Find the exact solution to the recurrence equation:
  \[ a_n = a_{n-1} + 2a_{n-2}, \text{ where } a_0 = 1 \text{ and } a_1 = 8 \]

• Theorem 9 tells us that if there are two real roots:
  \[ a_n = \alpha r_1^n + \beta r_2^n \text{ for } n \geq 0 \]
  \[ \text{where } a_0 = \alpha + \beta, \ a_1 = r_1\alpha + r_2\beta \]
  Since here \( r_1 = 2, r_2 = -1 \): \( a_n = \alpha 2^n + \beta n(-1)^n \)

• Plugging into equations for \( a_0 \) and \( a_1 \), solving for \( \alpha \) and \( \beta \):
  \[ 1 = \alpha + \beta \]
  \[ 8 = 2\alpha - \beta \]
  Add: \( 9 = 3\alpha \), so \( \alpha = 3, \beta = -2 \)

• Putting it all together: \( a_n = 3(2^n) - 2(-1)^n \)
Proof of correctness by induction on \( n \)

- Trying to prove: \( a_n = 3(2^n) - 2(-1)^n \)
  Given: \( a_n = a_{n-1} + 2a_{n-2}, \ a_0 = 1, \ a_1 = 8 \)
Proof of correctness by induction on $n$

- Trying to prove: $a_n = 3(2^n) - 2(-1)^n$
  
  Given: $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 1$, $a_1 = 8$

- Base case (does our equation work for $a_0$ and $a_1$?):
  
  $a_0 = 3(2^0) - 2(-1)^0 = 1$ ✔

  $a_1 = 3(2^1) - 2(-1)^1 = 8$ ✔
Proof of correctness by induction on $n$

- Trying to prove: $a_n = 3(2^n) - 2(-1)^n$
  
  Given: $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 1$, $a_1 = 8$

- Base case (does our equation work for $a_0$ and $a_1$?):
  
  $a_0 = 3(2^0) - 2(-1)^0 = 1$ ✔
  $a_1 = 3(2^1) - 2(-1)^1 = 8$ ✔

- Inductive step (does it work for $a_n$, when $a_n = a_{n-1} + 2a_{n-2}$?):

  $a_n = a_{n-1} + 2a_{n-2}$
  
  $= 3(2^{n-1}) - 2(-1)^{n-1} + 2(3(2^{n-2}) - 2(-1)^{n-2})$
  $= 6(2^{n-2}) - 4((-1)^{n-2}) + 3(2^{n-1}) - 2(-1)^{n-1}$
  $= 3(2^{n-1}) + 4((-1)^{n-1}) + 3(2^{n-1}) - 2(-1)^{n-1}$
  $= 6(2^{n-1}) + 2((-1)^{n-1})$
  $= 3(2^n) - 2((-1)^n)$ ✔
Example 2: Gambler’s Ruin

- A gambler repeatedly bets a flipped coin will come up heads.
  - If the coin is heads, the gambler wins $1.
  - If the coin is tails, the gambler loses $1.
  - If the gambler ever reaches $M he/she will stop.
Example 2: Gambler’s Ruin

• A gambler repeatedly bets a flipped coin will come up heads.
  – If the coin is heads, the gambler wins $1.
  – If the coin is tails, the gambler loses $1.
  – If the gambler ever reaches $M he/she will stop.

• Let $P_k$ = probability gambler loses all $k$ he/she has (= “ruin”)

$$P_k = P(H) \cdot P(\text{ruin}|H) + P(T) \cdot P(\text{ruin}|T)$$

$$P_k = \frac{1}{2} \cdot P(\text{ruin}|H) + \frac{1}{2} \cdot P(\text{ruin}|T) \quad \text{(since we’re flipping a coin)}$$

$$P_k = \frac{1}{2} \cdot P(\text{ruin}|\text{win }$1$) + \frac{1}{2} \cdot P(\text{ruin}|\text{lose }$1$)$$

$$P_k = \frac{1}{2} \cdot P_{k+1} + \frac{1}{2} \cdot P_{k-1}$$
Example 2: Gambler’s Ruin

- A gambler repeatedly bets a flipped coin will come up heads.
  - If the coin is heads, the gambler wins $1.
  - If the coin is tails, the gambler loses $1.
  - If the gambler ever reaches $M he/she will stop.

- Let \( P_k \) = probability gambler loses all $k he/she has (“ruin”)
  \[
  P_k = P(H) \cdot P(\text{ruin}|H) + P(T) \cdot P(\text{ruin}|T)
  \]
  \[
  P_k = \frac{1}{2} \cdot P(\text{ruin}|H) + \frac{1}{2} \cdot P(\text{ruin}|T) \quad \text{(since we’re flipping a coin)}
  \]
  \[
  P_k = \frac{1}{2} \cdot P(\text{ruin}|\text{win }$1) + \frac{1}{2} \cdot P(\text{ruin}|\text{lose }$1)
  \]
  \[
  P_k = \frac{1}{2} \cdot P_{k+1} + \frac{1}{2} \cdot P_{k-1}
  \]

If we manipulate the equation we can then apply Theorem 9…

\[
-\frac{1}{2} P_{k+1} = - P_k + \frac{1}{2} P_{k-1}
\]

\[
P_{k+1} = 2P_k - P_{k-1}
\]

\[
P_k = 2P_{k-1} - P_{k-2}
\]
Example 2: Gambler’s Ruin continued

• We just learned: $P_k = 2P_{k-1} - P_{k-2}$

• Recall: the characteristic equation for $a_n = ba_{n-1} + ca_{n-2}$ is $r^2 - br - c = 0$

• Here ours is:
  
  $r^2 - 2r + 1 = 0$

  $(r-1)(r-1) = 0$

  So we have the repeated root, $r = 1$
Example 2: Gambler’s Ruin continued

- We just learned: \( P_k = 2P_{k-1} - P_{k-2} \)

- Recall: the characteristic equation for \( a_n = ba_{n-1} + ca_{n-2} \) is
  \[ r^2 - br - c = 0 \]
- Here ours is:
  \[ r^2 - 2r + 1 = 0 \]
  \( (r-1)(r-1) = 0 \)
  So we have the repeated root, \( r = 1 \)

- We know \( P_0 = 1 \) (if we start with $0 we’re already ruined)
- We know \( P_M = 0 \) (if we start with $M we quit playing the game)
Example 2: Gambler’s Ruin continued

- We just learned: \( P_k = 2P_{k-1} - P_{k-2}, \ P_0 = 1, \ P_M = 0 \)

- Theorem 9 tells us that if there is one repeated real root, \( r \):
  \[ a_n = \alpha r^n + \beta nr^n \text{ for } n \geq 0 \]
  where: \( a_0 = \alpha \) and \( a_1 = r\alpha + r\beta \)

- Here \( r = 1 \), so \( a_n = \alpha + \beta n \), \( \alpha = 1 \) (we don’t have an \( a_1 \))
Example 2: Gambler’s Ruin continued

• We just learned: \( P_k = 2P_{k-1} - P_{k-2} \), \( P_0 = 1 \), \( P_M = 0 \)

• Theorem 9 tells us that if there is one repeated real root, \( r \):
  \[
  a_n = \alpha r^n + \beta n r^n \quad \text{for } n \geq 0
  \]
  where: \( a_0 = \alpha \) and \( a_1 = r\alpha + r\beta \)

• Here \( r = 1 \), so \( a_n = \alpha + \beta n \Rightarrow \alpha = 1 \) (we don’t have an \( a_1 \))

• Plugging into equation for \( a_n \), solving for \( \beta \):
  \[
  0 = \alpha + \beta M
  \]
  \[
  \Rightarrow 0 = 1 + \beta M
  \]
  \[
  \Rightarrow \beta M = -1
  \]
  \[
  \Rightarrow \beta = -(1/M)
  \]
Example 2: Gambler’s Ruin continued

- We just learned: \( P_k = 2P_{k-1} - P_{k-2}, P_0 = 1, P_M = 0 \)

- Theorem 9 tells us that if there is one repeated real root, \( r \):
  \[ a_n = \alpha r^n + \beta nr^n \text{ for } n \geq 0 \]
  where: \( a_0 = \alpha \) and \( a_1 = r\alpha + r\beta \)

- Here \( r = 1 \), so \( a_n = \alpha + \beta n \), \( \alpha = 1 \) (we don’t have an \( a_1 \))

- Plugging into equation for \( a_n \), solving for \( \beta \):
  \[ 0 = \alpha + \beta M, 0 = 1 + \beta M, \beta M = -1, \beta = -(1/M) \]

- Plugging into \( a_n \) again: \( P_n = 1 - n/M \)
Example 2: Gambler’s Ruin continued

- $P_n = 1 - n/M$, so our probability of ruin, $P_k = 1 - k/M$
Example 2: Gambler’s Ruin continued

- $P_n = 1 - n/M$, so our probability of ruin, $P_k = 1 - k/M$

- If we have $10 and won’t stop playing unless we have $100 what is the probability that we will lose all $10?

  - $k = 10$
  - $M = 100$
  - $P_{10} = 1 - 10/100 = .90$, or 90%!
Example 2: Gambler’s Ruin continued

• \( P_n = 1 - n/M \), so our probability of ruin, \( P_k = 1 - k/M \)

• If we have $10 and won’t stop playing unless we have $100, what is the probability that we will lose our $10?
  
  \[
  k = 10 \\
  M = 100 \\
  P_{10} = 1 - \frac{10}{100} = 0.90, \text{ or } 90\%!
  \]

• If we have $10 but only want to win $12, what is the probability that we will lose our initial $10?
  
  \[
  P_{10} = 1 - \frac{10}{12} = \frac{1}{6} = 0.1667, \text{ now only } 16.67\%
  \]
Problems 14

- P14.1 Prove, using mathematical induction, that a $2^n \times 2^n$ chessboard that is missing any single $1 \times 1$ square can always be tiled by L-shaped triominoes. (A triomino is shown at right.)
- P14.2 A unit circle contains seven points. Prove that there exists a pair of points separated by distance $\leq 1$.
- P14.3 Same as P14.2, but with six points.
DT Theorem 9

- Consider a recurrence of the form \( a_n = ba_{n-1} + ca_{n-2} \) for \( n \geq 2 \)

- Theorem 9 says that the form of the explicit solution depends on the roots of the characteristic polynomial \( r^2 = br + c \)

- If there are two distinct real roots \( r_1 \) and \( r_2 \), \( a_n = \alpha r_1^n + \beta r_2^n \)

- If there is one repeated real root \( r \), \( a_n = \alpha r^n + \beta nr^n \)
Complexity of DQ for Long Multiplication

• This slide: illustrate “unrolling” of recurrence or, “substitution”
  – First line of attack if no convenient theorem available…

• Multiply two 2s-digit numbers, \([wx] \cdot [yz]\)
  – 4 n/2-digit multiplications: \(xz, wz, xy, wy\)
  – Digit-shifting: multiplication by \(10^s, 10^{2s}\)
  – 3 additions

\[
T(n) = 4T(n/2) + \theta(n)
\]

• \(T(n) \leq 4 \ T(n/2) + cn\)
  \[
  \leq 4 \ [4T(n/4) + cn/2] + cn
  = 16 \ T(n/4) + (1 + 2)cn
  \leq 16 \ [4T(n/8) + cn/4] + (1 + 2)cn
  = 64 \ T(n/8) + (1 + 2 + 4)cn
  
  \ldots
  \]

  \[
  \leq 4^k \ T(n/2^k) + (1 + 2 + 4 + \ldots + 2^{k-1})cn
  \]

For \(k = \log_2{n}\): \(T(n) \leq n^2 \ T(1) + cn^2 = O(n^2)\)

\(O(n^2)\) makes sense
Recurrence  \( T(n) \leq a \cdot T(n/b) + O(n^d) \)

1)  \( T(n) = O(n^d) \) if \( a < b^d \)
2)  \( T(n) = O(n^d \log n) \) if \( a = b^d \)
3)  \( T(n) = O(n^{\log_b a}) \) if \( a > b^d \)

Type (3): long multiplication, matrix multiplication
Type (2): mergesort
Master Theorem Examples

• Mergesort $T(n) = 2T(n/2) + \Theta(n)$

\[
T(n) = \Theta(n \log n)
\]

• Matrix Multiply $T(n) = 8T(n/2) + \Theta(n^2)$

\[
T(n) = \Theta(n^{\log_2 8}) \approx \Theta(n^3)
\]
Proof of “Master Theorem”
CSE 101 material, don’t worry for now, but you’ll need this in a year

Recurrence: \( T(n) \leq a \cdot T(n/b) + O(n^d) \)

1) \( T(n) = O(n^d) \) if \( a < b^d \)
2) \( T(n) = O(n^d \log n) \) if \( a = b^d \)
3) \( T(n) = O(n^{\log_b a}) \) if \( a > b^d \)

- Assume \( n \) is a power of \( b \) \( \rightarrow \) can ignore rounding in \( \lceil n/b \rceil \)
- Subproblem size decreases by factor of \( b \) with each level of recursion \( \rightarrow \) reaches base case after \( \log_b n \) levels
- Branching factor \( a \) \( \rightarrow k^{th} \) level of tree has \( a^k \) subproblems each of size \( n/b^k \)
- Total work done at \( k^{th} \) level is \( a^k \times O(n/b^k)^d = O(n^d) \times (a / b^d)^k \)
Proof of “Master Theorem”

CSE 101 material, don’t worry for now, but you’ll need this in a year

Recurrence \[ T(n) \leq a \cdot T(n/b) + O(n^d) \]

1) \( T(n) = O(n^d) \) if \( a < b^d \)

2) \( T(n) = O(n^d \log n) \) if \( a = b^d \)

3) \( T(n) = O(n^{\log_b a}) \) if \( a > b^d \)

• Total work done at \( k^{\text{th}} \) level is \( a^k \times O(n/b^k)^d = O(n^d) \times (a / b^d)^k \)
• As \( k \) goes from 0 (root) to \( \log_b n \) (leaves), have geometric series with ratio \( a / b^d \)
  • \( a / b^d < 1 \rightarrow \) sum is \( O(n^d) \)
  • \( a / b^d > 1 \rightarrow \) sum is given by last term, \( O(n^{\log_b a}) \)
  • \( a / b^d = 1 \rightarrow \) sum is given by \( O(\log n) \) terms equal to \( O(n^d) \)
EXTRA