UCSD CSE 21, Spring 2014

Mathematics for Algorithms and System Analysis

Midterm Review

Class URL: http://vlsicad.ucsd.edu/courses/cse21-s14/
Review Session

• Get out a pencil and paper
• Lets solve problems as we go
• You will learn a lot more than you would if you just stared at this screen
Review Session

• Topics to be covered in first half:
  - Counting sequences given constraints
  - Multinomial problems
  - Random Walks
  - Expectation
Counting Sequences Given Constraints

• Some examples you have solved
  – HW #1, Question 4
    • Counting n digit numbers
  – HW #1, Question 8 (but solved by Stars and Bars)
    • A monotone increasing number
Counting Sequences Given Constraints

Suppose that license plates are seven alphanumeric characters long (0-9) and (A-Z). What is the probability that my randomly selected license plate ends with “ING” and two of the first four letters are vowels?

- Ex: HOOPING
- Ex: LEANING
- Ex: EAR2ING
Counting Sequences Given Constraints

• Suppose that license plates are seven alphanumeric characters long (0-9) and (A-Z). What is the probability that my randomly selected license plate ends with “ING” and exactly two of the first four letters are vowels?

  – The last three characters are all chosen.
  – We have to choose where the vowels go in the first 4. There are C(4,2) places for them to go.
  – Then within the left most list of 4 characters we have (31)(31)(5)(5) choices

  – Total probability: \( \frac{C(4,2)(31)(31)(5)(5)(1)(1)(1)}{36^7} \)
Expectation

• Suppose in some dystopian future, license plates become currency (suppose they all have seven characters), and that they are valued by the sum of the values of their characters: A-Z are given values 1-26 respectively, and 0-9 having values 0-9. If all alphanumeric combinations are possible, what is the expected future value of a random license plate?
Expectation

• How to get started?
  – First notice that all the characters are independent
  – Then calculate the expected value of an individual character
  – How to calculate expected value of an individual character?
  – You just need the probability and value of each outcome
Let’s find the expected value of one random character:

- 0 has probability 1/36
- 1-9 have probability 2/36 (from both 1-9 and A-I)
- 10-26 have probability 1/36 (from J-Z)

\[
\begin{align*}
&= \frac{1}{36}(0) + \frac{2}{36}(1) + \frac{2}{36}(2) + \ldots + \frac{2}{36}(9) + \frac{1}{36}(10) + \frac{1}{36}(11) + \ldots + \frac{1}{36}(26) \\
&= \frac{1}{18}(1+2+\ldots+9) + \frac{1}{36}(10+11+\ldots+26) = \frac{5}{2} + \frac{17}{2} = 11
\end{align*}
\]
Suppose in some dystopian future, license plates become currency, and that they are valued like by the sum of their characters with A-Z given values 1-26 respectively, and 0-9 having values 0-9. If all alphanumeric combinations are possible, what is the expected future value of a random license plate?

Let’s calculate the expected value of one character:

\[
= (1/18)(1+2+\ldots+9) + (1/36)(10+11+\ldots+26) = 5/2 + 17/2 = 11
\]

Then the average license plate has value “77”
**Expectation**

Then the average license plate has value “77”

Note that the maximum license plate has value \((7)(26) = 182\)

So the plates are skewed towards lower values because the probability of values 1-9 is doubled!
Multinomial / Binomial Problems

• Some examples you have solved
  - HW #1 Question 5
  - HW #2 Question 4
  - HW #4 Question 2
  - HW #4 Question 10
Multinomial / Binomial Problems

• Example:
  - What is the coefficient of $w^3x^4y^2z^3$ in the expansion of $(w+2x+3y+z)^{12}$ by the multinomial theorem?
Multinomial / Binomial Problems

• How to get started?
  – You need to first calculate the multinomial coefficient. This is the bulk of the work in this problem.
  – Then you need to multiply it by the appropriate powers of the coefficients.
Multinomial / Binomial Problems

• What is the coefficient of $w^3x^4y^2z^3$ in $(w+2x+3y+z)^{12}$?

$$(w+2x+3y+z)^{12} = \sum_{k_1+k_2+k_3+k_4=12} \binom{12}{k_1,k_2,k_3,k_4} w^{k_1}(2x)^{k_2}(3y)^{k_3}z^{k_4}$$

– Fill in the term inside the sum with:

  • $k_1 = 3$, $k_2 = 4$, $k_3 = 2$, $k_4 = 3$
Multinomial / Binomial Problems

• What is the coefficient of $w^3x^4y^2z^3$ in the expansion of $(w+2x+3y+z)^{12}$ by the multinomial theorem?

\[- \binom{12}{3,4,2,3} 2^4 3^2\]
Multinomial / Binomial Problems

• How many anagrams of flibbertigibbet are there?
Multinomial / Binomial Problems

• How many anagrams of flibbertigibbet are there?
  - How to get started?
  - First, there are 15! ways to rearrange the letters.
  - Then you have to divide out the ways which the identical (indistinguishable) occurrences of individual letters occur.
Multinomial / Binomial Problems

• How many anagrams of flibbertigibbet are there?
  – Count the number of letters: 15 total
  – F: 1 count
  – L: 1 count
  – I: 3 counts
  – B: 4 counts
  – E: 2 counts
  – R: 1 count
  – T: 2 counts
  – G: 1 count

• Then the number of anagrams is:

\[
\binom{15}{3,4,2,2} = \frac{15!}{3!4!2!2!}
\]
Multinomial / Binomial Problems

• How many ways can we form 4 teams from 12 people, so that each team has 3 members?
Multinomial / Binomial Problems

• How many ways can we form 4 teams from 12 people, so that each team has 3 members?
  – We get started on this by thinking about how many ways we can arrange the 12 people, and dividing out by the number of rearrangements of the people within the teams.
Multinomial / Binomial Problems

• How many ways can we form 4 teams from 12 people, so that each team has 3 members?

• This is another multinomial coefficient (ordered set partition) problem and (part of) the answer is \( \binom{12}{3,3,3,3} \)

• What is the other part?
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• Basically, the teams don’t have names or any other distinguishing features except who the team members are. So we have to “unaccount” for ordering. Now what is the answer?
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• Basically, the teams don’t have names or any other distinguishing features except who the team members are. So we have to “unaccount” for ordering. Now what is the answer?
• \( \binom{12}{3,3,3,3} / 24 \)
Random Walk

• A random walk is a mathematical formalization of a path that consists of a succession of random steps.
  
  [Link to Wikipedia article on random walks](http://upload.wikimedia.org/wikipedia/commons/f/f3/Random_walk_2500_animated.svg)

• Y value is random walk on the integers

![Graph of random walk](image-url)
Random Walk

• Suppose a very lazy crab is on a 1-dimensional beach. Every minute it either steps 1 unit to the right with probability 2/3, or 1 unit to the left, with probability 1/3.
  
  – If there is a really delicious dead fish 10 steps to the right, what is the probability that the crab lands on the fish for the first time at 10, 11, 12 minutes from now? A wave is going to wash it away at t=13 minutes.
Random Walk

- If there is a really delicious dead fish 10 steps to the right, what is the probability that the crab lands on the fish for the first time at 10, 11 or 12 minutes from now? A wave is going to wash it away at t=13 minutes.
- 10 minutes: This can only be \((2/3)^{10}\)
- 11 minutes: This is probability 0. The crab could hit the fish at t=10 but then it would have to step away. So the number of net right steps can only be an odd number in this case.
Random Walk

- If there is a really delicious dead fish 10 steps to the right, what is the probability that the crab lands on the fish for the first time at 10, 11 or 12 minutes from now? A wave is going to wash it away at t=13 minutes.
- 12 minutes: \( \left( \frac{2}{3} \right)^{11} \left( \frac{1}{3} \right) \) will get us there but in how many ways? And how many times did we land on the fish twice?
- 12 ways (choose where the left step goes)
- And 2 of them landed on the fish at t = 10.
- \( 12 \left( \frac{2}{3} \right)^{11} \left( \frac{1}{3} \right) - 2 \left( \frac{2}{3} \right)^{11} \left( \frac{1}{3} \right) = 10 \left( \frac{2}{3} \right)^{11} \left( \frac{1}{3} \right) \)