CSE 21: Mathematics for Algorithms and Systems Analysis
Week 5 Discussion – Extreme Midterm Edition

David Lisuk

April 30, 2014
1. Announcements
2. Midterm Issues
3. Theory Review - Sterling Numbers vs Stars and Bars
4. Review Homework 4
5. Random Variables/Expectation/Variance
6. IF TIME: Questions About Midterm

iClicker Frequency: BA
Announcements

- Midterm Thursday (ABK) and Friday (RRR) – Goto the right one
- ABK – Review Section Slides on Website (We’ll go over some of the examples today)
- RRR – Review Section Thursday (5/1) CENTER 119 from 7-9pm
- Homework 5 due Monday 5/5 at midnight

iClicker Frequency: BA
Topics on ABK’s Midterm

- Counting sequences given constraints
- Multinomial problems
- Distributing items to sets (Stars/Bars and Sterling Numbers)
- Poker problems
- Expectation
- Probability of sequences
- Random walk probability
- Geometric probability
- Function counting
Topics on RRR’s Midterm

- Counting sequences given constraints
- Multinomial problems
- Distributing items to sets (Stars/Bars, Sterling numbers, Bell Numbers)
- Poker problems
- Probability of sequences
- Function counting
- Function Composition/Inverse
- Random graphs
- Permutations/Derangements
- Substitution ciphers
- Plugboard wires/Enigma
- Expectation
- Inclusion Exclusion Principle
Write your name and PID at the top of EVERY page (each separate piece of paper)
Midterm Directions – Both

- Write your name and PID at the top of EVERY page (each separate piece of paper)
- Please note that there are problems on both the front and back of every page (including the front)
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- Additional scratch paper is available at the front of the class
Common problem is to distribute items to set
Distributing Items to Sets

- Common problem is to distribute items to set
- Use stars/bars or sterling numbers of second kind
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- Use stars/bars or sterling numbers of second kind

<table>
<thead>
<tr>
<th></th>
<th>Stars and Bars</th>
<th>Sterling Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sets</td>
<td>Distinguishable</td>
<td>Indistinguishable</td>
</tr>
<tr>
<td></td>
<td>Can be empty</td>
<td>Must be non empty</td>
</tr>
<tr>
<td>Objects</td>
<td>Indistinguishable</td>
<td>Distinguishable</td>
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Sterling Numbers of the Second Kind

- Divides a set of $n$ distinguishable objects into $k$ indistinguishable non empty subsets ($k$ partitions)

A Sterling number of the second kind counts how to do this. A table of Sterling numbers will be provided, or just keep it as $S(n, k)$. 

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Bell numbers are a way to enumerate all partitions of $n$ distinguishable objects into any number of sets.
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All partitions of 5 distinguishable objects

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Distribution Example

After going to the animal shelter you came home with 10 cats. On the way home you bought 15 different toys and 20 treats to give them. How many ways are there to distribute the toys such that every cat gets at least 1 treat and the toys are distributed among 5 cats?
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### iClicker Question (Frequency: BA)

What is the number of ways to distribute toys?

(a) $S(15, 5)$
(b) $S(15, 5)P(10, 5)$
(c) $S(15, 5)C(10, 5)$
(d) $C(15 + 5 - 1, 5 - 1)$
(e) $C(15 + 5 - 1, 5 - 1)P(10, 5)$
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(c) $S(10, 10)$
(d) $C(10 + 10 - 1, 10 - 1)$
(e) $C(10 + 10 - 1, 10 - 1)10!$
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- Total functions $B \rightarrow G$ is $3^5$
- Onto functions $B \rightarrow G$ is $S(5, 3)3!$
- Probability of choosing an onto function from all possible functions $\frac{S(5, 3)3!}{3^5}$
Question: What is the maximum number of edges possible in a random graph with 4 vertices?
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Count $\sum_{i=1}^{4} 4 - i = 3 + 2 + 1 = 6$ (see board)
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- Count $\sum_{i=1}^{4} 4 - i = 3 + 2 + 1 = 6$ (see board)
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Question: What is the probability that a random graph $G(10, 0.6)$ has 0 edges?
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- $(1 - 0.6)^{10 \choose 2}$
iClicker Question (Frequency: BA)

What is the probability that a random graph $G(10, 0.6)$ has 12 edges? \( \binom{10}{2} = 45 \)

(a) \( \frac{\binom{45}{12}}{2^{45}} \)

(b) \( 0.6^{12} \cdot 0.4^{45-12} \)

(c) \( \binom{45}{12} \cdot 0.6^{12} \cdot 0.4^{45-12} \)

(d) \( \frac{2 \times 47}{\binom{47}{2}} \)

(e) \( \frac{2 \times 47/2!}{\binom{47}{2}} \)
What is the probability that a random graph $G(10, 0.6)$ has 12 edges? ($\binom{10}{2} = 45$)

(a) $\frac{\binom{45}{12}}{2^{45}}$

(b) $0.6^{12}0.4^{45-12}$

(c) $\binom{45}{12}0.6^{12}0.4^{45-12} \star$

(d) $\frac{2\times47}{C(47,2)}$

(e) $\frac{2\times47/2!}{C(47,2)}$
A variable $x$ is called a random variable if it takes on values following some probability distribution.

Example: Slot Machine

<table>
<thead>
<tr>
<th>Value</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.990</td>
</tr>
<tr>
<td>1</td>
<td>0.009</td>
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<td>10</td>
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Random Variables

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**Example: Slot Machine**

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</tr>
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- A *Uniform* distribution is one where all possible choices have equal probability.
- These are analyzed with *expectation* and *variance*.
Expectation

- Think about an average

\[ \{ x_1, x_2, x_3, \ldots, x_n \} \]

- Average value

\[ \frac{1}{n} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{1}{n} x_i \]
Expectation

- Think about an average

\[ \{x_1, x_2, x_3, \ldots, x_n\} \]

- Average value \( \frac{1}{n} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{1}{n} x_i \)

- Can be thought of as a probability distribution over \( x_i \) with \( \frac{1}{n} \) probability that each \( x \) is chosen
Expectation

- Think about an average
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- Then average is
  \[ \sum_{i=1}^{n} P(x_i)x_i \]
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This is called the expected value

\[ E(X) = \sum_{x \in X} P(x) x \]
**iClicker Question (Frequency: BA)**

Given the above distribution what is the expected payout of the slot machine?

(a) \( \frac{1}{3}(-1 + 10 + 1000) \)

(b) \( \frac{1}{1009}(-1 + 10 + 1000) \)

(c) \(-0.99 + 0.09 + 1 = 1.9 \)

(d) \(0.99 + 0.009 + 0.001 = 1 \)
Expectation Example

Slot Machine

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Variance captures how much you expect a random variable’s value to be spread out.
Variance

- Variance captures how much you expect a random variable’s value to be spread out
- \( \text{Var}(x) = \mathbb{E}[(x - \mathbb{E}(x))^2] \), with \( \mathbb{E}(x) \) be constant
Variance captures how much you expect a random variable’s value to be spread out.

\[ \text{Var}(x) = \mathbf{E}[(x - \mathbf{E}(x))^2], \text{ with } \mathbf{E}(x) \text{ be constant} \]

Will not be on either midterm.

Pretty much compute the same way as expected value, but just subtract the expected value from all payouts.
**Slot Machine** Compute the variance of the slot machine below:

<table>
<thead>
<tr>
<th>$P(x)$</th>
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<th>$(x - E(x))^2$</th>
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<tbody>
<tr>
<td>0.990</td>
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<td>8.4</td>
</tr>
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<td>65.6</td>
</tr>
<tr>
<td>0.001</td>
<td>1000</td>
<td>996203</td>
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$$\text{Var}(x) = 0.990 \times 8.4 + 0.009 \times 65.6 + 0.001 \times 996203$$
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$\text{Var}(x) = 0.990 \times 8.4 + 0.009 \times 65.6 + 0.001 \times 996203$
Good Luck!

Any Questions?