Additional Examples and Notes

For a sample of \( r \) objects from \( n \) objects:

\[
\begin{array}{c|c|c}
\text{ordered} & \text{without replacement} & \text{with replacement} \\
\hline
nPr &= \frac{n!}{(n-r)!} & n^r \\
\hline
nCr &= \binom{n}{r} = \frac{n!}{r!(n-r)!} & \frac{n-1+r}{r} \\
\end{array}
\]

1. Find the number of different third order partial derivatives of a continuous function of three variables, \( f(x, y, z) \) This is an unordered sample of \( r \) objects from \( n \) with replacement (since the order of differentiation does not matter): \( \binom{n-1+r}{r} = \binom{3-1+3}{3} = 10 \)

In fact we can list all of the possibilities:

\[
\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} = f_{xyz}, f_{xxx}, f_{yyy}, f_{zzz}, f_{xxz}, f_{yyz}, f_{yzx}, f_{xyy}, f_{xzz}
\]

For second order or higher partial derivatives all with the same variable, like \( f_{xxx} \) are called pure. The others, like \( f_{xyy} \) and \( f_{yzx} \) are called mixed. If we ask how many different mixed third order partial derivatives of a function of three variables, then we subtract the number of pure third order partials (in this case 3) to get \( 10 - 3 = 7 \).

2. Poker. A poker hand is defined as drawing five cards from a deck of 52 standard playing cards without replacement. In the game of poker, the following hands are possible; they are listed in increasing desirability and decreasing probability. In the hand definitions, the word value refers to: A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2. This sequence also describes the relative ranks of the cards (except that an Ace may count as a 1 for the purposes of making a straight). For each of the 13 values, there are exactly 4 cards with that value, one of each suit \( \spadesuit, \diamondsuit, \clubsuit, \heartsuit \). (i.e. there are precisely four 7's in a deck: \( 7\spadesuit, 7\diamondsuit, 7\clubsuit, 7\heartsuit \).)

(i) nothing: \( 3\clubsuit, 7\diamondsuit, 8\heartsuit, 6\spadesuit, J\spadesuit \)
(ii) one pair: \( 5\clubsuit, 5\diamondsuit, K\heartsuit, Q\spadesuit, J\spadesuit \)
(iii) two pair: \( 8\spadesuit, 8\heartsuit, K\spadesuit, Q\heartsuit, Q\spadesuit \)
(iv) three of a kind: \( 10\heartsuit, 10\spadesuit, 10\diamondsuit, K\spadesuit, Q\spadesuit \)
(v) straight (not straight flush): \( 7\spadesuit, 6\heartsuit, 5\diamondsuit, 4\clubsuit, 3\heartsuit \)
(vi) flush (not straight flush): \( 8\spadesuit, 7\spadesuit, 5\heartsuit, 4\heartsuit, 3\heartsuit \)
(vii) full house: \( 9\clubsuit, 9\heartsuit, 9\spadesuit, 4\diamondsuit, A\spadesuit \)
(viii) four of a kind: \( 5\heartsuit, 4\clubsuit, 3\spadesuit, 2\heartsuit, A\heartsuit \)
(ix) straight flush: \( A\heartsuit, K\spadesuit, Q\heartsuit, J\heartsuit, 10\heartsuit \)
(x) royal flush: \( A\spadesuit, K\spadesuit, Q\diamondsuit, J\diamondsuit, 10\spadesuit \)
(i) We must have all five cards with different values while avoiding a straight in $\binom{13}{5} - 10$ ways. Then, we assign any suit to the five cards while avoiding a flush, we can do this in $4^5 - 4$ ways.

$P(\text{nothing}) = \frac{\left(\binom{13}{5} - 10\right) \cdot (4^5 - 4)}{\binom{52}{5}} \approx .501177$

(ii) First we choose the value (13 ways) for the pair, then assign suits to those two cards ($4^2$ ways), then pick three values (different from the value of the first two) for the remaining two cards ($12^3$ ways), and then assign suits to those cards ($4^3$ ways).

$P(\text{one pair}) = \frac{13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3}{\binom{52}{5}} \approx .422569$

(iii) First we choose the two values for the two pairs ($\binom{13}{2}$ ways), then assign suits to the first pair ($4^2$ ways), and assign suits to the second pair ($4^2$ ways); finally pick the value of the remaining card (11 ways) and assign a suit (4 ways)

$P(\text{two pairs}) = \frac{\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 11 \cdot 4}{\binom{52}{5}} \approx .047539$

(iv) First we choose the value (13 ways) for the three of a kind, then assign suits to those three cards ($4^3$ ways), then pick two values (different from the value of the first 3) for the remaining two cards ($12^2$ ways), and then assign suits to those cards ($4^2$ ways).

$P(\text{3 of a kind}) = \frac{13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4^2}{\binom{52}{5}} \approx .021128$

(v) First we choose the sequence of five in a row, which must start with a 5 and then descend, \{5,4,3,2,A\}, \{6,5,4,3,2\}, ..., \{K,Q,J,10,9\}, \{A,K,Q,J,10\}, (10 ways) then assign suits to those five cards $4^5 - 4$ ways

$P(\text{straight, not straight flush}) = \frac{10 \cdot (4^5 - 4)}{\binom{52}{5}} \approx .003925$
(vi) First we choose the values \((\binom{13}{5} \text{ ways})\) for the five cards, then assign the same suit to those five cards in 4 ways.

\[
P(\text{flush, not straight flush}) = \frac{4 \left( \binom{13}{5} - 10 \right)}{\binom{52}{5}} \approx 0.001965
\]

(vii) First we choose the value \((13 \text{ ways})\) for the three of a kind, then assign suits to those three cards \((\binom{4}{3} \text{ ways})\), then pick the value for the pair \((12 \text{ ways})\), and assign suits \((\binom{4}{2} \text{ ways})\) to those cards.

\[
P(\text{full house}) = \frac{13 \left( \binom{4}{3} \right) \cdot 12 \left( \binom{4}{2} \right)}{\binom{52}{5}} \approx 0.001441
\]

(viii) First we choose the value \((13 \text{ ways})\) for the four of a kind, then assign suits to those four cards \((\binom{4}{4} \text{ ways})\), then pick the last card from the 12 remaining values, and assign a suit \((4 \text{ ways})\) to the last card.

\[
P(4 \text{ of a kind}) = \frac{13 \left( \binom{4}{4} \right) \cdot 12 \left( \binom{4}{2} \right)}{\binom{52}{5}} \approx 0.000240
\]

(ix) There are 9 ways to have 5 values that will make a straight (that is not a royal flush), and 4 ways to pick the suit

\[
P(\text{straight flush}) = \frac{9 \cdot 4}{52 \choose 5} \approx 0.0001385
\]

(x) There is one way to have 5 values that will make a special straight called a royal flush, and 4 ways to pick the suit

\[
P(\text{royal flush}) = \frac{1 \cdot 4}{52 \choose 5} \approx 0.00001539
\]
Two evenly matched teams A and N play a best of 5 series of games (the team who wins three games wins the series). What is the probability that the series winner was never behind. For example ANAA is not possible because after the third game A was behind 2 games to 1, but AANNA is a possible outcome.

The series will either last 3, 4 or 5 games, and there are \(2\binom{3}{2} + 2\binom{4}{2} + 2\binom{5}{2} = 20\) ways for a best of 5 game series to be played (see 1.3-11). For the winner to never be behind, we want to stay on one side of the dotted line in the center. There is one way to do this if the series lasts 3 games, 2 ways for 4 games AANA or ANAA, and 2 ways for 5 games: AANNA or ANANA. The probability of the winner never being behind is \(\frac{2(1 + 2 + 2)}{20} = \frac{10}{20} = 0.5\).
Two evenly matched teams $A$ and $N$ play a best of 7 series of games (the team who wins four games wins the series). What is the probability that the series winner was never behind. For example $ANNA$ is not possible because after the third game $A$ was behind 2 games to 1, but $AAANNNA$ is a possible outcome.

The series will either last 4, 5, 6 or 7 games, and there are $2 \left( \begin{array}{c} 3 \\ 3 \end{array} \right) + 2 \left( \begin{array}{c} 4 \\ 3 \end{array} \right) + 2 \left( \begin{array}{c} 5 \\ 3 \end{array} \right) + 2 \left( \begin{array}{c} 6 \\ 3 \end{array} \right) = 70$ ways for a best of 7 game series to be played (see 1.3-11). For the winner to never be behind, we want to stay on one side of the dotted line in the center. There is one way to do this if the series lasts 4 games; 3 ways for 5 games: $AANAA$, $ANAAA$ or $AAANA$; 5 ways for 6 games: $AAANNA$, $AANNAA$, $AANAAN$ or $ANANAA$ and 5 ways for 7 games: $AAANNNA$, $AANANNA$, $AANNNNA$, $ANAANNA$ or $ANANANA$ The probability of the winner never being behind is:

$$\frac{2(1 + 3 + 5 + 5)}{70} = \frac{28}{70} = 0.4$$