1. A valid license plate must be made of 7 characters, coming from the set of 26 letters \{A, B, \ldots, Z\} and 10 digits \{0, 1, \ldots, 9\}. All of the characters must be distinct and at most 2 letters can be used. How many valid license plates are there?

2. How many five-digit numbers exist such that
   a. No digit appears more than once?
   b. The number is divisible by 5?
   c. The number is even?
   d. Exactly three of its digits are even?

   Notice that a number cannot have a leading digit of 0.

3. How many rearrangements are there of the letters in the word LOLLIPOP?

4. Charles has twelve identical tennis balls, five identical bones, and two identical dog toys that he wants to distribute to his five dogs. If every dog must receive at least one tennis ball and two dog toys cannot be sorted to the same dog, how many ways can Charles distribute the items?

5. There are N nickels and 3 dimes in a coin collection. How many ways can the coins be stacked such that every dime is touching a nickel? Give your answer in terms of N.

6. What is the coefficient of \(x^3y^2\) in the expansion of \((3x + 2y)^5\)?

7. If \(f = (12)(34)(56)\), \(g = (135)(246)\), and \(h = (654321)\) are three permutations in cycle form, what is the cycle form of
   a. \(f \circ g \circ h\)?
   b. \(h \circ g \circ f\)?
   c. \(g \circ g \circ f\)?
   d. \(h \circ f \circ h\)?

8. In how many ways can Oliver place ten distinctly colored balls into five distinctly colored urns if each urn can only hold two balls?

9. Jane randomly removes three cards from a standard deck of cards. What is the probability that
   a. They form a triple, i.e., three of a kind?
   b. They are all the same color?
c. They form a pair and a single?
d. They form three singles of different suits?

10. Every week, Billy buys a lottery ticket for $10 with a one-in-a-billion chance at winning $100,000,000 and a one-in-a-hundred chance at winning $900. What is his expected return

a. After 10 weeks?
b. After 10 weeks if he buys 3 lottery tickets every week?

11. An ant travels one unit to the left or one unit to the right every turn with equal probability. What is the chance that the ant is at its starting point after

a. Two turns?
b. Four turns?
c. Six turns?
d. 2N turns?

12. CHALLENGE: A beetle, starting at the origin on a coordinate plane, travels one unit upwards or rightwards every turn with equal probability. After 2N turns, the beetle is at location (N, N). What is the probability that the beetle has never once been at a location (x, y) such that y > x?

13. What is the probability that in a room of seven people,

a. No two have the same birthday (assuming there are 365 equally likely, possible birthdays)?
b. Some two have the same birthday?

14. A total of five people are at a social event. If initially nobody has met anybody, in how many orders can the people meet each other if everyone is to meet everyone else?

15. Marissa has three $10 bills, four $50 bills, and one $100 bill in her pocket. If she just ordered a five-gallon tub of boba for $110 at the local tea silo, what is the probability that she can pay for her purchases (possibly receiving change in return) if

a. She draws two bills out at random?
b. She draws three bills out at random?
c. She draws four bills out at random?
d. She draws five bills out at random?

16. A team of N technicians has an N / (N + 2) chance of fixing a router. If the company dispatches at random 1, 2, or 3 technicians to fix a router, what is the probability that

a. The router gets fixed on the first dispatch of technicians?
b. The router gets fixed in no more than two dispatches of technicians?
17. John and Sarah play games of Rock, Paper, Scissors in which each randomly chooses Rock, Paper, or Scissors at every turn. Rock beats Scissors, which beats Paper, which in turn beats Rock. What is the expected number of games they play

a. If they play until one person wins?
b. If they play until John wins?
c. If they play until Rock, Paper, and Scissors have each been chosen at least once?

18. In how many ways can five boys and five girls sit in a row if each must sit next to at least one person of the same gender?

19. In how many ways can five boys and five girls sit in a circle such that no two boys sit next to each other? Notice that two arrangements that are obtainable from each other by rotation are considered to be the same arrangement.

20. John tosses a coin five times. If he gets heads, he moves a marker two units to the right. If he gets tails, he moved the marker one unit to the left. If the marker is initially at the center of a circular platform of radius five units, what is the probability that the marker falls off of the circular platform as a result of the five-move sequence?

21. In order to get past pipes in his game of Flappy Bird, John found that he has to tap his phone twice in every two-second window (i.e., twice in [0,2 seconds), twice in [2,4 seconds), etc.) with the second tap being no less than 0.3 seconds after the first tap. If John taps two times at random in every two-second window, what is

   a. The probability that John gets past the first pipe?
   b. John’s expected score, i.e., number of pipes he gets past?
   c. John’s expected score if the second tap must be after the 1.5-second mark in every two-second window?

22. CHALLENGE: Suppose John plays a different version of Flappy Bird and must tap his phone three times every three seconds, with the difference in time between successive taps being no less than 0.3 seconds in order to get past a pipe. If he taps three times at random in every three-second window, what is his new expected score?

23. A wedding photographer arranges six people in a row for a photograph from a group of 12 people (six from each side, including the bride and groom). How many ways are there to arrange the people if … (consider each of the following restrictions separately):
   a. Both the bride and the groom must be in the picture?
   b. The bride and groom must be next to each other?
   c. The groom must be positioned somewhere to the right of the bride?
   d. There must be an equal number of people in the picture from the bride’s and groom’s families?
24. Consider six numbered circles and three lettered squares. In how many ways can the numbered circles be joined to the lettered squares, so that every circle is joined to exactly one square and: (consider each of the following separately)
   a. there is no other restriction?
   b. every square is joined to at least one circle?
   c. every square is joined to two circles?

25. Suppose that five different balls are placed into four labeled boxes at random.
   a. What is the probability that no box is empty?
   b. What is the probability that exactly one box is empty?
   c. What is the probability that at least one box is empty?

26. A box contains 10 light bulbs, 5 of which are bad and 5 of which are good. If 4 light bulbs are drawn at random without replacement from the box, what is the expected number of good light bulbs drawn?

27. How many different decimal numbers (strings of symbols 0, 1, …, 9) of length 8 either start with a 9 or end with a 00?

28. Suppose there are 10 balls in an urn: four blue, four red, and two green. The balls are also numbered 1 to 10. How many ways are there to select an ordered sample of four balls without replacement, such that there are two blue balls and two red balls in the sample?

29. A player is dealt four spade cards from an ordinary deck of 52 cards. If five more cards are given to the player, what is the probability that none of them are spades?

30. Six cards are dealt (without replacement) from an ordinary deck of 52 cards. Let X denote the number of Hearts that are dealt, and let Y denote the number of face cards (Jack, Queen, or King) dealt. What is the expected value of X + Y?

31. For 40 weeks, once per hour during the 40-hour work week, an employee of Best Cars draws a ball from an urn that contains 1 black and 9 white balls. If black is drawn, a $10 bill is tacked to a bulletin board. At the end of the 40 weeks, the money is given to charity. What is the expected amount of money given?

32. Assume that n > 2k. In how many ways can you divide n identical chocolates among k children such that each child receives at least two chocolates?

33. It's late on Halloween and four people show up at your door: two dressed as superheroes and two dressed as villains. How many ways are there to distribute your remaining 15 pieces of candy (5 identical chocolates, 6 identical lollipops, and 4 identical jawbreakers) to the trick-or-treaters so that each superhero gets at least one piece of candy, and each villain gets no chocolate?
34. How many functions \( f \) are there from \( \{1, \ldots, n\} \) to \( \{-1, 0, 1\} \) such that there exists at least one \( i \in \{1, \ldots, n\} \) for which \( f(i) = 0 \)?

35. Of 28 people surveyed, 18 drive foreign cars and 10 drive domestic cars. If five of these people are selected at random, what is the probability that at least three of them drive foreign cars?

UCSD CSE 21 Spring 2014, Section B00

ABK Things To Know, 140421 DRAFT / VERY PRELIMINARY!!!

- **Sets and lists**
  - Set: collection of distinct objects where order does not matter
  - List (or string): ordered collection
    - (Whether repetition is allowed will be specified when referring to a list)
  - Size or cardinality: \( k \)-set (\( k \)-list) is a set (list) of size \( k \)
  - Theorem 1: There are \( n^k \) ways to form a \( k \)-list from elements of an \( n \)-set.
    - (Order matters, with repetition)
  - Theorem 4: There are \( n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = n! / (n-k)! = P(n,k) \) ways to form a \( k \)-list from distinct elements of an \( n \)-set.
    - (Order matters, without repetition)
  - If order does not matter \( \Rightarrow \) get “subsets”, “selection”/“choosing” instead of “lists”, “ordering”

- **Counting**
  - Theorem 2: Rule of Product
    - Set \( S \) has \( 2^{|S|} \) subsets
  - Cartesian product: Given sets \( C_1, C_2, \ldots, C_k \), their Cartesian product is \( C_1 \times C_2 \times \ldots \times C_k \) and has elements \( (x_1,x_2,\ldots,x_k) \) where \( x_i \in C_i, \ i = 1, \ldots, k \)
    - The cardinality of the Cartesian product \( |C_1 \times C_2 \times \ldots \times C_k| = |C_1| \cdot |C_2| \cdot \ldots \cdot |C_k| \)
    - Lexicographic order = dictionary order
  - Theorem 3: Rule of Sum
  - Basic ideas in counting
    - How to count “constructively”
      - When to add == apply Rule of Sum
      - When to multiply == apply Rule of Product
    - When ordering matters vs. when it doesn’t matter
      - (permutations vs. combinations)
    - When to correct for over-counting
      - (e.g., when there are symmetries)
    - How to break down into cases
    - How to deal with a “restriction”
    - When to count the complement
      - (when it’s easier to do that)
• PIE = Principle of Inclusion-Exclusion
  \[ |A \cup B| = |A| + |B| - |A \cap B| \]

• Theorem 6: Algebraic rules for sets
  \[
  \text{E.g., DeMorgan: } (A \cap B)^c = A^c \cup B^c \quad ; \quad (A \cup B)^c = A^c \cap B^c
  \]

• Theorem 7: Combinations. \( C(n,k) = \text{number of ways to select k out of n elements} = \frac{n!}{[(n-k)! k!]} \)
  \[ \text{Recurrence: } C(n,k) = C(n-1,k) + C(n-1,k-1) \quad // \text{understand why this is true!} \]

• Binomial Theorem:
  \[
  \begin{align*}
  (1+x)^n &= C(n,0)x^0 + C(n,1)x^1 + C(n,2)x^2 + \ldots + C(n,n)x^n \\
  (a+b)^n &= C(n,0)a^n b^0 + C(n,1)a^{n-1}b^1 + \ldots + C(n,n)a^0 b^n 
  \end{align*}
  \]

• Putting \( n \) objects into \( k \) marked boxes
  \[ \text{Given } n \text{ indistinguishable objects and } k \text{ marked boxes, there are } C(n + k - 1, n) = C(n + k - 1, k - 1) \text{ ways to distribute the objects among the boxes.} = \text{Order} \]

  **Form Method**
  \[ \text{Also: "staircase method"} \]
  \[ \text{“Counting with replacement, when order does not matter”} \]
  \[ \text{How many integer solutions are there to the equation } x_1 + x_2 + x_3 + x_4 = 12, \text{ with } x_i \geq 0? \]
  \[ \text{How many different orders for six hot dogs are possible if there are three varieties of hot dog?} \]
  \[ \text{How many ways are there to distribute 20 (identical) sticks of licorice among five children?} \]
  \[ \text{Distributing } n \text{ distinct objects into } k \text{ marked boxes is equivalent to lining up the objects in a row and stamping one of the } k \text{ different box names on each object. There are } k \cdot k \cdot \ldots \cdot k = k^n \text{ ways to do this.} \]
  \[ \text{Given } n \text{ distinct objects and } k \text{ marked boxes, for } m_1 + m_2 + \ldots + m_k = n \text{ there are } (n; m_1,m_2,\ldots,m_k) = n! / (m_1!m_2!\ldots m_k!) \text{ ways to put } m_i \text{ objects into the } i^{\text{th}} \text{ box for all } i = 1, \ldots, k. = \text{Multinomial} \]
  \[ \text{(CL, Example 18)} \]
  \[ \text{Rearrangements of GOOGLE: 6 slots for letters = 6 objects; 4 boxes G, O, L, E for letters; } m_1 = 2, m_2 = 2, m_3 = 1, m_4 = 1. \]

• Probability
  \[ \text{Sample space } U = \text{universal set of possible outcomes} \]
  \[ \text{Event } E = \text{collection of possible outcomes, } E \subseteq U \]
  \[ \text{Pr(union of events)} \leq \text{sum of probabilities of the individual events} \]
  \[ \text{Pr(union of events)} = \text{sum of probabilities of the events when the events are disjoint} \]
  \[ \text{Pr}(A \text{ and } B) = \text{Pr}(A) \cdot \text{Pr}(B) \text{ when } A, B \text{ independent} \]
  \[ \text{Pr}(A) = 1 - \text{Pr}(A^c) \quad \text{Pr(event)} = 1 - \text{Pr(event’s complement)} \]
  \[ \text{Expected trials up to and including first success} = 1 / \text{Pr(success)} \]

  **Example problems**
  \[ \text{Given an urn with 5 red, 5 black balls, you draw two balls without replacement.} \]
  \[ \text{What is the probability that you have drawn one black and one red ball?} \]
  \[ \text{What is the probability of obtaining exactly 3 heads in 5 tosses of a fair coin?} \]
  \[ \text{What is the probability of obtaining exactly 3 heads in 5 tosses of a biased coin for which } \text{Pr(heads)} = 1/4? \]
What is the probability of obtaining exactly 3 A’s, 2 B’s and 1 C in 6 spins of a 3-segment roulette wheel, for which \( \text{Pr}(A) = 1/10, \text{Pr}(B) = 3/5 \text{ and Pr}(C) = 3/10 \)?

in 5 tosses of a fair coin?

Functions

If A and B are sets, a function from A to B is a rule that tells us how to find a unique b \( \in B \) for each a \( \in A \). We write \( f:A \to B \) to indicate that \( f \) is a function from \( A \) to \( B \).

- A = domain of \( f \)
- B the range (or, codomain) of \( f \)
- To specify a function completely, you must give its domain, range and rule.

- \( f \) is an injection or \( f \) is injective or \( f \) is 1-to-1 : \( f(x) = f(y) \Rightarrow x = y \). I.e., “\( f \) does not map two different elements of the domain to the same element of the range” (or, “nobody in the range gets hit twice”).

- \( f \) is a surjection or \( f \) is surjective or \( f \) is onto : for every element \( y \) in the range of \( f \), there exists some element \( x \) in the domain of \( f \) such that \( f(x) = y \). I.e., “everybody in the range gets hit”.

- \( f \) is a bijection or \( f \) is bijective or \( f \) defines a 1-to-1 correspondence between its domain and its range if \( f \) is both injective and surjective (1-to-1 and onto).