1. What is the maximum possible number of vertices in a binary tree of height \( h \)? The height of a binary tree is the length of the longest path from the root vertex to any vertex in the tree. 

\[
[2^{h+1} - 1]
\]

**Solution** Testing with small values we guess that the answer is \( 2^{h+1} - 1 \). We will prove this by induction. When \( h = 0 \) this is indeed true as there is just one root vertex. Now suppose it is true for \( h = k \). Now we create a tree of height \( k + 1 \) by joining two trees of height \( k \) through a root node. This gives the total number of vertices = \( 2^k - 1 + 2^k - 1 + 1 = 2^{k+1} - 1 \) which is what we wanted.

2. Perform DFS on the following directed graph starting at vertex A. How many trees are present in the DFS search forest that is obtained?  

[B]

\[
\begin{align*}
A \rightarrow B \rightarrow C \\
D \rightarrow E \rightarrow F \\
G \rightarrow H \rightarrow I
\end{align*}
\]

A. 0  
B. 1 (correct)  
C. 2  
D. 3

**Solution** All the vertices of the given graph are reachable from vertex A. As a result, performing DFS starting from A will result in only one DFS tree, which contains all the vertices of the graph.
3. We want to count monotone paths between points with integer coordinates. Only two kinds of steps are allowed: a right-step which increments the x coordinate and an up-step which increments the y coordinate. Suppose that there is an obstacle at (5, 4) such that no path can pass through this point. How many monotone paths are there from (0, 0) to (9, 6)? [Note: Monotone path is simply a "staircase" path that only uses rightward and upward edges to get from the source to the destination.]

\[ C(15, 6) - C(9, 5) \times C(6, 4) = 3115 \]

Solution A path from origin to (9, 6) will require 9 right steps and 6 up steps, giving a total of \( C(15, 6) \) paths from origin to (9, 6). The number of paths that pass through the point (5, 4) are the paths that first reach (5, 4) and then go to (9, 6). (5, 4) can be reached from origin in \( C(9, 5) \) ways and (9, 6) can be reached from (5, 4) in \( C(6, 4) \) ways. The number of paths passing through (5, 4) is thus \( C(9, 5) \times C(6, 4) \). The required number of paths is therefore \( C(15, 6) - C(9, 5) \times C(6, 4) \).

4. Suppose an undirected graph has 13 vertices and 8 of the vertices have degree 2 each. What could be a possible value of the sum of degrees of the other five vertices? (Hint: the handshake lemma)

A. 10 (correct)
B. 5
C. 15
D. 25

Solution Apply the handshake lemma to see that the sum of the degrees of all vertices of the graph must be even. Thus, as eight of the nodes have degree two each, then the sum of the degrees of the remaining five vertices must be even too. The only possible option is 10. Hence option A is correct.

5. In a connected graph containing \( N \) vertices, there is exactly one path between any two vertices. The minimum possible number of edges in this graph, and the maximum number of edges in this graph, are respectively:

\[ D \]

A. \( \Theta(N) \) and \( \Theta(N^2) \)
B. \( \Theta(1) \) and \( \Theta(N^2) \)
C. \( \Theta(1) \) and \( \Theta(N) \)
D. \( \Theta(N) \) and \( \Theta(N) \) (correct)

Solution The given condition is one of the if and only if conditions for a graph \( G \) to be a tree. Hence \( G \) is a tree and has \( N - 1 \) edges. If we define functions \( f(N) = N - 1 \) and \( g(N) = N \), then we can say that \( f(N) = \Theta(g(N)) = \Theta(N) \).
6. Let \( f(n) = 2^n \) and \( g(n) = n! \). Which of the following best represents the relationship between \( f(n) \) and \( g(n) \)? (Hint: take logs)  

A. \( f(n) = O(g(n)) \)  
B. \( f(n) = \Theta(g(n)) \)  
C. \( f(n) = \Omega(g(n)) \) (correct)  

Solution

Let’s consider the ratio \( \frac{\log(f(n))}{\log(g(n))} \) as \( n \to \infty \).

\[
\frac{\log(f(n))}{\log(g(n))} = \frac{3n \log(2)}{\log(n!)}  
\]

\[
3n \log(2) = \Theta(n)  
\]

\[
\frac{\Theta(n)}{\Theta(n \log n)} = 0  
\]

Thus \( f(n) = \Omega(g(n)) \).

7. Given an arbitrary, connected, undirected and unweighted (i.e., all edge costs = 1) graph \( G = (V, E) \), which of the following will always find the shortest path from a given starting vertex \( s \in V \) to any other vertex \( t \in V \)?  

A. DFS starting from \( s \)  
B. BFS starting from \( s \) (correct)  
C. Both (A) and (B)  
D. Neither (A) nor (B)  

Solution

DFS (DPV textbook, p96) may find the shortest path from \( s \) to \( t \), but it is not guaranteed and there are counterexamples showing how DFS fail to find a shortest path.

This counterexample shows DFS find a path \( A \rightarrow B \rightarrow C \rightarrow E \) first, therefore it fails to find a shortest path \( A \rightarrow D \rightarrow E \).
8. The tightest time complexity of DFS in a graph \( G = (V, E) \), which holds for ANY graph \( G \), is:

A. \( O(|V| + |E|) \) (correct)
B. \( O(|V| \cdot |E|) \)
C. \( O(|V|^2) \)
D. \( O(|E|^2) \)

**Solution** The time complexity of DFS is \( O(|V| + |E|) \), which follows directly from the procedure as analysed in Section 3.2.2 of the DPV textbook.

9. For the pseudo-code given below, select the correct asymptotic running time in \( \Theta \) notation:

```plaintext
for i = 1 to n do
    j = i
    while j < n do
        j = j + 4
    end while
end for
```

A. \( \Theta(n^2) \) (correct)
B. \( \Theta(<log(n)> \)
C. \( \Theta(\sqrt{n}) \)
D. \( \Theta(n^3) \)

(Credit: Berkeley CS 170 website)

**Solution** In the given pseudo-code, line 2 and the condition in while are ‘unit time’ operations in the outer loop which is executed \( n \) times. Line 4 is another ‘unit time’ operation that is within the inner loop which is executed \( n/4 \) times for each iteration of the outer loop. Line 4 is thus executed a total of \( n \cdot (n/4) \) times. The total number of unit-time operations is \( 2n + n(n/4) = n^2/4 + 2n = O(n^2) \). Since \( n^2/4 + 2n \) is also \( \Omega(n^2) \), the correct answer is \( \Theta(n^2) \).

10. After running DFS on a directed graph, a vertex \( v \) has \textit{pre} label 8 and \textit{post} label 12. A vertex \( u \) has \textit{pre} label 13 and \textit{post} label 17. If \( (u, v) \) is an edge of the graph, then it is a:

A. Back edge
B. Forward edge
C. Cross edge (correct)
D. Tree edge

**Solution** It can be seen that the *pre* and *post* numbers of vertices $v$ and $u$ do not overlap; i.e. $[8, 12]$ and $[13, 17]$ are disjoint intervals. Hence, the edge from $u$ to $v$, $(u, v)$, corresponds to a Cross edge of the graph.