These "model solutions" are provided to give an idea of the level of detail and formality that would receive full credit in homework writeups or exams.

1 Algorithm design.

Exercise 3.7.

A bipartite graph is a graph $G = (V; E)$ whose vertices can be partitioned into two sets ($V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$) such that there are no edges between vertices in the same set (for instance, if $u, v \in V_1$, then there is no edge between $u$ and $v$). Give a linear-time algorithm to determine whether an undirected graph is bipartite

Solution

Algorithm. Our algorithm will recursively partition the vertices of $G$ into two sets and check that they maintain the two-coloring property of a bipartite graph as it proceeds. If a vertex is found that violates the coloring property we immediately halt and return false. If we completely explore the graph without finding such a violation we return true.

Let $\{-1, 1\}$ be the 2 colors. Then visited[$u$] will be 0 if $u$ has not been visited yet, 1 if $u$ has been colored with the color 1 and $-1$ if it has been with the color $-1$.

Input: graph $G = (V, E)$
Output: true if the graph $G$ is bipartite, false otherwise

procedure isBipartite(G):
    for each $v$ in $V$:
        visited[$v$] = 0
    for each $v$ in $V$:
        if explore(G, v, 1) == 0:
            return false
    return true
end

procedure explore(G, v, color):
    visited[$v$] = color
    for each edge $(v, u)$ in $E$:
        if visited[$u$] <> 0 & visited[$u$] + color <> 0:
            return false
        else
            explore(G, u, color * (-1))
end
Running time. Our algorithm mimics DFS on a graph with a constant amount of additional work to check for a valid coloring of the graph for each edge. Therefore the running time of our algorithm is the same as DFS, $O(|V| + |E|)$.

Proof of correctness. A graph is bipartite if and only if a valid two-coloring of the graph exists. Clearly, our algorithm will find a two-coloring if one exists since any alternating of the coloring of connected vertices will be a valid two-coloring of a bipartite graph.

Now, consider the case when a valid two-coloring does not exist (i.e. the graph is not bipartite). Then, there must exist some cycle such that node $u$ of the cycle is colored with one color of $\{1, -1\}$ and node $v$ is colored with the other color and there is an edge connecting $v$ and $u$. When our algorithm reaches node $v$ it will check the edge connecting $v$ and $u$, see the conflict and return false.

2 Proofs and counterexamples.

Exercise 3.6.

In an undirected graph, the degree $d(u)$ of a vertex $u$ is the number of neighbors $u$ has, or equivalently, the number of edges incident upon it. In a directed graph, we distinguish between the indegree $d_{\text{in}}(u)$, which is the number of edges into $u$, and the outdegree $d_{\text{out}}(u)$, the number of edges leaving $u$.

(a) Show that in an undirected graph, $\sum_{u \in V} d(u) = 2|E|$.

(b) Use part (a) to show that in an undirected graph, there must be an even number of vertices whose degree is odd.

(c) Does a similar statement hold for the number of vertices with odd indegree in a directed graph?

Solution

(a) Direct proof. For a vertex $u$, $d(u)$ is the number of edges incident upon it. So $\sum_{u \in V} d(u)$ is the count of all edges incident on all the vertices. As an edge has exactly two vertices, all edges are counted exactly twice. So we have $\sum_{u \in V} d(u) = 2|E|$.

(b) Proof by contradiction. Suppose that there are an odd number of vertices with an odd degree. Then the sum of all degrees is odd. Yet, we proved in (a) that the sum of all degrees is even (as $2|E|$ must be an even number). This contradicts our assumption. So there must be an even number of vertices with an odd degree.

(c) Proof by counterexample. This statement does not hold in a directed graph when considering indegrees. For example, in the 2 vertex simple graph $(A \rightarrow B)$, we have $d_{\text{in}}(A) = 0$ and $d_{\text{in}}(B) = 1$. So, we have only 1 (odd) vertex with an odd degree.

3 Proof by induction

Example 1

Prove that it is possible to color the regions formed by any number of lines in the plane using only two colors.

Solution

We will prove the above statement by weak induction.

Base case. Let $n$ be the number of lines in the plane. Clearly the statement is true when $n = 1$. We use one color for left (or top) side of the line, and the other color for the right (or bottom) side of the line.
**Inductive hypothesis.** Now, assume as our inductive hypothesis that we can color the regions formed by \( n \) lines in the plane where \( n > 1 \) with only two colors.

**Inductive step.** Now, consider the case when we have \( n + 1 \) lines. First, we remove one line and note that by our inductive hypothesis we can color all regions with only two colors. Next, we add back the removed line. On only one side of the line, we swap all the colors. We now have \( n + 1 \) lines in the plane and a possible two-coloring since either:

- A region is split by the newly added line in which case we flipped the color for only half of the region.
- Or a region is not split by the line in which case we may have flipped the color of the region or kept it the same. Since all neighboring regions on the same side of the line were also flipped or kept the same this change does affect the correctness of the coloring.

So, by induction we have show that \( \forall n > 0 \) it is possible to color the regions formed by any number of lines in the plane using only two colors.

**Note: Weak induction**

As our inductive hypothesis was only concerned with using a value of \( n \) to show the claim holds for a value of \( n + 1 \) we used weak induction. A proof by strong induction would have required us to assume the claim true for any number of lines \( k \) such that \( 1 < k < n \) in order to show the statement holds for \( n \) lines.

**Example 2**

Prove that if \( n \) is a natural number and \( 1 + x > 0 \), then \( (1 + x)^n \geq 1 + nx \).

**Solution**

We will prove the claim by induction on \( n \).

**Base case.** When \( n = 0 \) the claim holds since \( (1 + x)^0 \geq 1 + 0 \).

**Inductive hypothesis.** Now, assume as our inductive hypothesis that \( (1 + x)^n \geq 1 + nx \) for some value of \( n > 0 \).

**Inductive step.** Now, can show the following chain of inequalities:

\[
(1 + x)^{n+1} = (1 + x)^n(1 + x) \\
\geq (1 + nx)(1 + x) \quad \text{by our inductive hypothesis} \\
\geq 1 + nx + x + nx^2 \\
\geq 1 + (n + 1)x + nx^2 \\
\geq 1 + (n + 1)x \\
\]

\[nx^2 \geq 0 \text{ since } n \geq 0\]

So, by induction, we have shown that \( \forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx \).